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Conservation Laws for Highly Dispersive Optical Solitons in Birefringent Fibers

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Abstract—This paper reports conservation laws for highly dispersive optical solitons in birefringent fibers. Three forms of nonlinearities are studied which are Kerr, polynomial and nonlocal laws. Power, linear momentum and Hamiltonian are conserved for these types of nonlinear refractive index.

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1. INTRODUCTION

One of the most innovative extensions of optical soliton dynamics is the study of highly dispersive (HD) optical solitons that was first conducted during 2019, and later a plethora of papers reported a wide range of results [1–19]. Typically, this concept of HD solitons appears when, in addition to the usual chromatic dispersion (CD), the effects from intermodal dispersion (IMD), third-order dispersion (3OD), fourth–order dispersion (4OD), fifth-order dispersion (5OD) and sixth-order dispersion (6OD) are taken into account. These additional dispersion effects would produce pronounced soliton radiation. However, these effects are neglected in the studies and attention is paid only to the discrete spectrum of the scattering data. Very recently, the study of HD solitons has been extended to eighth-order dispersion along with eighth-order nonlinearity of the governing model [17].

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The main governing model to study HD solitons is the nonlinear Schrödinger's equation (NLSE) that includes these six dispersion terms along with some form of nonlinearity. HD optical solitons with NLSE as its governing model has been successfully studied in polarization-preserving fibers with four forms of nonlinearity. They are Kerr law, polynomial law, quadratic-cubic law and nonlocal law. In these cases, soliton solutions have been established and conservation laws are also secured [1–10]. Later, the perturbed NLSE with HD solitons, both in the presence and absence of self-phase modulation effects, have been studied with the aid of semi-inverse variational principle [11, 12]. Some introductory results on HD solitons in birefringent fibers have also been reported in [17].

The present paper continues the journey further along. It is devoted to the retrieval of conservation laws for HD solitons in birefringent fibers that serves as an extension to the results in polarization-preserving fibers. The method of multipliers [20–25] has been implemented to secure the conserved densities of the model in birefringent fibers for three forms of nonlinearity. The case of QC nonlinearity has been discarded since it is not possible to obtain them using this multiplier method. Subsequently, the reported bright soliton solutions are utilized to obtain the conserved quantities, for three nonlinear forms, from those reported conserved densities. The details are all pen-pictured in the rest of the manuscript.

1.1. Governing Model

The NLSE that models highly dispersive optical solitons in a polarization-preserving optical fiber is written as

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + F\left(|q|^2\right) q = 0, \tag{1.1}$$

where $q(x,t)$ is a complex-valued function that represents the wave profile. The independent variables x and t are spatial and temporal coordinates, respectively, and $i = \sqrt{-1}$. The real-valued coefficients a_j for $1 \leqslant j \leqslant 6$ represent intermodal dispersion (IMD), chromatic dispersion (CD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (4OD) and fifth-order dispersion (5OD), respectively. Finally, the functional F accounts for the nonlinear form of refractive index and

$$F\left(|q|^2\right) q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n) \times (-m,m)\, ; R^2\right).$$

For birefringent fibers, Eq. (1.1) splits into vector-coupled NLSEs that have the following structure:

$$iu_t + ia_1^1 u_x + a_2^1 u_{xx} + ia_3^1 u_{xxx} + a_4^1 u_{xxxx} + ia_5^1 u_{xxxxx} + a_6^1 u_{xxxxxx} + G\left(|u|^2, |v|^2\right) u = 0, \tag{1.2}$$

$$iv_t + ia_1^2 v_x + a_2^2 v_{xx} + ia_3^2 v_{xxx} + a_4^2 v_{xxxx} + ia_5^2 v_{xxxxx} + a_6^2 v_{xxxxxx} + H\left(|u|^2, |v|^2\right) u = 0, \tag{1.3}$$

where the functionals G and H emanate from self–phase modulation (SPM) and cross–phase modulation (XPM) effects. Also, the real-valued coefficients a_j^l for $1 \leqslant j \leqslant 6$ represents IMD, CD, 3OD, 4OD, 5OD and 6ODm, respectively, along the two components of birefringent fibers for $l = 1, 2$. The conservation laws for Eqs. (1.2) and (1.3) will be now derived for four forms of nonlinear refractive index after an introductory discussion on these laws.

2. PRELIMINARIES ON CONSERVATION LAWS

The role and methods associated with conservation laws are now well established and there have been some momentous works in these areas recently, building on the contributions made by Noether which generally dealt with variational problems, those that admit variational symmetries. It is not surprising then that much of the recent works focused on generalizations as far as constructions of conservation laws go, possibly nonvariational and preferably independent of a knowledge of symmetries.

A vast amount and extensively cited works are due to Anco & Bluman in [20, 21], inter alia, Anderson [22, 23], Kara & Mahomed [24]. The first of these deals extensively with the notion of 'multipliers' that if a differential equation times a factor (differential function) is closed, then the Euler operator annihilates this product so that finding conserved flows amounts to finding the factors. It turns out that the multipliers are solutions of the adjoint equation. Of course, one still needs to determine the corresponding conserved flows using, amongst others, homotopy formulae [25].

Consider an rth-order system of partial differential equations (pdes) of n independent variables $\underline{s} = (s_1, s_2, \dots, s_n)$ and m dependent variables $u = (u_1, u_2, \dots, u_m)$ viz.,

$$E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0, \qquad u = 1, \dots, \tilde{m}, \tag{2.1}$$

where a locally analytic function $f(\underline{s}, u, u_1, \dots, u_k)$ of a finite number of dependent variables $u, u_1, \dots, u_k$ denotes the collections of all first- , second- ,..., kth-order partial derivatives and s is a multivariable. That is,

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j D_i(u^\alpha), \dots \tag{2.2}$$

respectively, with the total differentiation operator with respect to s^i given by

$$D_i = \frac{\partial}{\partial s^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots \qquad i = 1, \dots, m. \tag{2.3}$$

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well-known result that the Euler–Lagrange operator annihilates a total divergence. Firstly, if $(T^{s1}, T^{s2}, \dots)$ is a conserved vector corresponding to a conservation law, then

$$D_{s1} T^{s1} + D_{s2} T^{s2} + \dots = 0 \tag{2.4}$$

along the solutions of the differential equation $E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0$. Moreover, if there exists a nontrivial differential function Q, called a “multiplier”, such that

$$Q(\underline{s}, u, u_{(1)} \dots) E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = D_{s1} T^{s1} + D_{s2} T^{s2} + \dots, \tag{2.5}$$

for some (conserved) vector $(T^{s1}, T^{s1}, \dots)$, then

$$\frac{\delta}{\delta u}[Q(\underline{s}, u, u_{(1)} \dots) E(\underline{s}, u, u_{(1)}, \dots, u_{(r)})] = 0, \tag{2.6}$$

where $\frac{\delta}{\delta u}$ is the Euler operator. Hence, one may determine the multipliers using (2.6) and then construct the corresponding conserved vectors; several approaches for this exist of which the better known one is the “homotopy” approach. If the system of differential equations is derived from a variational principle, then the conserved vector components are obtainable from Noether’s theorem, which requires, firstly, the construction of variational symmetries (vector fields) $X = \xi^{s^i} \frac{\partial}{\partial s^i} + \eta^{u^\alpha} \frac{\partial}{\partial u^\alpha}$ that leave the action integral invariant. It is well known that the vector fields that leave the system of differential equations invariant (generators of Lie point symmetries) contain the algebra of variational symmetries if the latter exists. Conservation laws may be expressed as conserved forms [23]. For example, if $\underline{s} = (t, x)$, the conserved form would be

$$\omega = T^t \mathrm{d}x - T^x \mathrm{d}t$$

(where (T^t, T^x) is the conserved vector such that $D_t T^t + D_x T^x = 0$ on the solutions of the PDE $E(x, t, u, u_{(1)}, \dots, u_{(r)}) = 0$). Here, T^t leads to the “conserved density” if t and x are time and space, respectively.

3. KERR LAW

The NLSE for highly dispersive optical solitons with the Kerr law of refractive index along polarization-preserving optical fiber is written as

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + b\,|q|^2\, q = 0. \tag{3.1}$$

The constant b is the coefficient of the Kerr law nonlinearity. For birefringent fibers, this model splits up into two components as follows:

$$iu_t + ia_1^1 u_x + a_2^1 u_{xx} + ia_3^1 u_{xxx} + a_4^1 u_{xxxx} + ia_5^1 u_{xxxxx} + a_6^1 u_{xxxxxx} + \left(b_1^1 |u|^2 + b_2^1 |v|^2\right) u = 0, \tag{3.2}$$

$$iv_t + ia_1^2 v_x + a_2^2 v_{xx} + ia_3^2 v_{xxx} + a_4^2 v_{xxxx} + ia_5^2 v_{xxxxx} + a_6^2 v_{xxxxxx} + \left(b_1^2 |v|^2 + b_2^2 |u|^2\right) v = 0 \tag{3.3}$$

after neglecting the effects of four-wave mixing (4WM). The constants b_j^l for $l = 1, 2$ represent SPM coefficients for $j = 1$ and XPM coefficients for $j = 2$.

Bright soliton solutions to (3.2) and (3.3) are [17]

$$\begin{aligned} u(x,t) = \Bigg\{ &\pm \sqrt{\frac{20160 a_6^1 \delta^2}{b_1^1 + b_2^1}} \operatorname{sech}\left[x - \begin{pmatrix} 5a_5^1 {\kappa_1}^4 - 6\, a_6^1 {\kappa_1}^5 - 2\, a_2^1 \kappa_1 \\ -3a_3^1 {\kappa_1}^2 + 4a_4^1 {\kappa_1}^3 + a_1^1 \end{pmatrix} t\right] \\ &\pm \sqrt{\frac{20160 a_6^1}{b_1^1 + b_2^1}} \operatorname{sech}^3\left[x - \begin{pmatrix} 5a_5^1 {\kappa_1}^4 - 6\, a_6^1 {\kappa_1}^5 - 2\, a_2^1 \kappa_1 \\ -3a_3^1 {\kappa_1}^2 + 4a_4^1 {\kappa_1}^3 + a_1^1 \end{pmatrix} t\right]\Bigg\} e^{i(-\kappa_1 x + w_1 t + \zeta_1)}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} v(x,t) = \Bigg\{ &\pm \sqrt{\frac{20160 a_6^2 \delta^2}{b_1^2 + b_2^2}} \operatorname{sech}\left[x - \begin{pmatrix} 5a_5^2 {\kappa_2}^4 - 6\, a_6^2 {\kappa_2}^5 - 2\, a_2^2 \kappa_2 \\ -3a_3^2 {\kappa_2}^2 + 4a_4^2 {\kappa_2}^3 + a_1^2 \end{pmatrix} t\right] \\ &\pm \sqrt{\frac{20160 a_6^2}{b_1^2 + b_2^2}} \operatorname{sech}^3\left[x - \begin{pmatrix} 5a_5^2 {\kappa_2}^4 - 6\, a_6^2 {\kappa_2}^5 - 2\, a_2^2 \kappa_2 \\ -3a_3^2 {\kappa_2}^2 + 4a_4^2 {\kappa_2}^3 + a_1^2 \end{pmatrix} t\right]\Bigg\} e^{i(-\kappa_2 x + w_2 t + \zeta_2)}. \end{aligned} \tag{3.5}$$

In (3.4) and (3.5), κ_j and ω_j are the frequencies and wave numbers along the two components of birefringent fibers, while ζ_j are the respective phase constants for $j = 1, 2$.

3.1. Conservation Laws

In the system above, we let $u = p + iw$ and $v = q + iz$ so that the system splits into a system of four PDEs whose conserved flows (T^t, T^x) are constructed using the multiplier approach. It turns out that for arbitrary values of the parameters we have a single multiplier $Q = (-p, -q, w, z)$ giving rise to the conserved vector

$$\begin{aligned} T^t &= -\frac{1}{2}(p^2 + w^2 + q^2 + z^2), \\ T^x &= -p_{xxxx} a_6^1\, w_x + q_{xxx} z a_4^2 - z_{xx} z a_3^2 - q_{xx} a_4^2\, z_x + p_{xxx} a_5^1\, p_x - z_{xxxx} z a_5^2 \\ &- z_x q a_2^2 + p_{xxx} a_6^1\, w_{xx} - w_{xxx} a_6^1\, p_{xx} - p_{xx} a_4^1\, w_x - z_{xxx} q a_4^2 - q_{xxxx} a_6^2\, z_x \\ &+ q_{xxx} a_6^2\, z_{xx} - w_{xxxxx} p a_6^1 + p_{xxx} w a_4^1 + z_{xxxx} a_6^2\, q_x - q_{xx} q a_3^2 - w_{xxx} p a_4^1 + w_{xxx} a_5^1\, w_x \\ &- p_{xxxx} p a_5^1 + q_{xxx} a_5^2\, q_x - z_{xxxxx} q a_6^2 - w_x p a_2^1 + q_x z a_2^2 + w_{xxxx} a_6^1\, p_x - p_{xx} p a_3^1 \\ &+ q_{xxxxx} z a_6^2 - w_{xxxx} w a_5^1 + z_{xx} a_4^2\, q_x + p_x w a_2^1 - z_{xxx} a_6^2\, q_{xx} + w_{xx} a_4^1\, p_x - q_{xxxx} q a_5^2 \\ &+ z_{xxx} a_5^2\, z_x + p_{xxxxx} w a_6^1 - w_{xx} w a_3^1 - \frac{1}{2} z^2 a_2^2 - \frac{1}{2} q^2 a_2^2 - \frac{1}{2} p^2 a_1^1 - \frac{1}{2} w^2 a_1^1 \\ &- \frac{1}{2} a_5^2\, {z_{xx}}^2 - \frac{1}{2} a_5^1\, {w_{xx}}^2 - \frac{1}{2} a_5^2\, {q_{xx}}^2 - \frac{1}{2} a_5^1\, {p_{xx}}^2 \\ &+ \frac{1}{2} a_3^2\, {z_x}^2 + \frac{1}{2} a_3^1\, {w_x}^2 + \frac{1}{2} a_3^2\, {q_x}^2 + \frac{1}{2} a_3^1\, {p_x}^2. \end{aligned} \tag{3.6}$$

Here, a corresponding power density of the complex system is

$$\Phi^t = |u|^2 + |v|^2.$$

If $b_2^1 = b_2^2$, then momentum and total power are conserved, giving rise to the following conserved densities T^t and/or fluxes T^x:

Linear momentum

$$\begin{aligned}
T^t &= -\Big[-\frac{1}{2}q_x z - \frac{1}{2}p_x w + \frac{1}{2}z_x q + \frac{1}{2}w_x p\Big],\\
T^x &= -z_{xxx}a_4^2\, z_x + w_{xxxx}a_6^1\, w_{xx} - \frac{1}{2}a_6^2\, {z_{xxx}}^2 - \frac{1}{2}a_6^1\, {w_{xxx}}^2 - \frac{1}{2}a_6^2\, {q_{xxx}}^2 + \frac{1}{2}qz_t + \frac{1}{2}pw_t\\
&+ \frac{1}{2}a_4^1\, {p_{xx}}^2 - \frac{1}{2}a_2^2\, {z_x}^2 - \frac{1}{2}a_2^1\, {w_x}^2 + \frac{1}{2}a_4^2\, {z_{xx}}^2 + \frac{1}{2}a_4^1\, {w_{xx}}^2 + \frac{1}{2}a_4^2\, {q_{xx}}^2\\
&- \frac{1}{2}a_2^2\, {q_x}^2 - \frac{1}{2}a_2^1\, {p_x}^2 - \frac{1}{2}zq_t - \frac{1}{2}wp_t - \frac{1}{2}a_6^1\, {p_{xxx}}^2 - p_{xxxx}a_5^1\, w_x + q_{xxxx}a_6^2\, q_{xx}\\
&- z_{xxxxx}a_6^2\, z_x - w_{xxx}a_5^1\, p_{xx} + p_{xxxx}a_6^1\, p_{xx} - q_{xxxx}a_5^2\, z_x - q_{xxx}a_4^2\, q_x + w_{xxxx}a_5^1\, p_x\\
&+ q_{xxx}a_5^2\, z_{xx} - w_{xxxxx}a_6^1\, w_x + w_{xx}a_3^1\, p_x + z_{xx}a_3^2\, q_x - w_{xxx}a_4^1\, w_x - p_{xxx}a_4^1\, p_x + z_{xxxx}a_5^2\, q_x\\
&+ z_{xxxx}a_6^2\, z_{xx} - p_{xxxxx}a_6^1\, p_x + p_{xxx}a_5^1\, w_{xx} - q_{xx}a_3^2\, z_x - q_{xxxxx}a_6^2\, q_x - z_{xxx}a_5^2\, q_{xx} - p_{xx}a_3^1\, w_x\\
&- \frac{1}{2}z^2 b_2^2\, w^2 - \frac{1}{2}z^2 b_2^2\, p^2 - \frac{1}{2}w^2 b_1^1\, p^2 - \frac{1}{2}z^2 b_1^2\, q^2\\
&- \frac{1}{4}b_1^1\, w^4 - \frac{1}{4}b_1^2\, q^4 - \frac{1}{2}q^2 b_2^2\, p^2 - \frac{1}{4}b_1^2\, z^4 - \frac{1}{2}w^2 b_2^2\, q^2 - \frac{1}{4}b_1^1\, p^4.
\end{aligned} \tag{3.7}$$

Momentum density:

$$\Phi^t = \mathcal{I}(u^* u_x) + \mathcal{I}(v^* v_x).$$

Hamiltonian:

$$\begin{aligned}
T^t &= \frac{1}{2}za_5^2\, q_{xxxxx} + \frac{1}{2}wa_6^1\, w_{xxxxxx} + \frac{1}{2}pa_4^1\, p_{xxxx} + \frac{1}{2}pa_6^1\, p_{xxxxxx} + \frac{1}{2}pa_2^1\, p_{xx} - \frac{1}{2}qa_2^1\, z_x\\
&+ \frac{1}{2}qa_6^2\, q_{xxxxxx} + \frac{1}{2}va_4^2\, q_{xxxx} - \frac{1}{2}pa_1^1\, w_x - \frac{1}{2}pa_3^1\, w_{xxx} + \frac{1}{2}wa_4^1\, w_{xxxx} - 1/2\, pa_5^1\, w_{xxxxx}\\
&+ \frac{1}{2}za_2^2\, z_{xx} + \frac{1}{2}qa_2^2\, q_{xx} - \frac{1}{2}qa_5^2\, z_{xxxxx} + \frac{1}{2}wa_2^1\, w_{xx} + \frac{1}{2}wa_5^1\, p_{xxxxx} + \frac{1}{2}wa_1^1\, p_x\\
&+ \frac{1}{2}za_6^2\, z_{xxxxxx} + \frac{1}{2}za_3^2\, q_{xxx} + \frac{1}{2}za_2^1\, q_x - \frac{1}{2}qa_3^2\, z_{xxx} + \frac{1}{2}za_4^2\, z_{xxxx} + \frac{1}{2}wa_3^1\, p_{xxx}\\
&+ \frac{1}{2}z^2 b_2^2\, w^2 + \frac{1}{2}z^2 b_2^2\, p^2 + \frac{1}{2}w^2 b_1^1\, p^2 + \frac{1}{2}z^2 b_1^2\, q^2\\
&+ \frac{1}{4}b_1^1\, w^4 + \frac{1}{4}b_1^2\, v^4 + \frac{1}{2}q^2 b_2^2\, p^2 + \frac{1}{4}b_1^2\, z^4 + \frac{1}{2}w^2 b_2^2\, q^2 + \frac{1}{4}b_1^1\, p^4,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\Phi^t = \frac{1}{2}\big[&-a_1^1\mathcal{I}(u^* u_x) - a_3^1\mathcal{I}(u^* u_{xxx}) - a_5^1\mathcal{I}(u^* u_{xxxxx}) - a_1^2\mathcal{I}(v^* v_x) - a_3^2\mathcal{I}(v^* v_{xxx}) - a_5^2\mathcal{I}(v^* v_{xxxxx})\\
&+ a_2^1\mathcal{R}(uu^*_{xx}) + a_4^1\mathcal{R}(uu^*_{xxxx}) + a_6^1\mathcal{R}(uu^*_{xxxxxx}) + a_2^2\mathcal{R}(vv^*_{xx}) + a_4^2\mathcal{R}(uu^*_{xxxx})\\
&+ a_6^2\mathcal{R}(uu^*_{xxxxxx}) \quad + \frac{1}{2}b_1^1|u|^4 + \frac{1}{2}b_1^2|v|^4 + b_2^2|u|^2|v|^2\big].
\end{aligned}$$

Therefore, the conserved quantities are

$$P = \int_{-\infty}^{\infty}\Big(|u|^2 + |v|^2\Big)\,dx = \frac{86}{15B}\left(A_1^2 + A_2^2\right), \tag{3.9}$$

$$M = 2i\int_{-\infty}^{\infty}\{(u^* u_x - uu^*_x) + (v^* v_x - vv^*_x)\}\,dx = \frac{86}{15B}\left(\kappa_1 A_1^2 + \kappa_2 A_2^2\right) \tag{3.10}$$

and

$$H = ia_1^1\int_{-\infty}^{\infty}(uu^*_x - u^* u_x)\,dx - a_2^1\int_{-\infty}^{\infty}(uu^*_{xx} + u^* u_{xx})\,dx - ia_3^1\int_{-\infty}^{\infty}(uu^*_{xxx} - u^* u_{xxx})\,dx$$

$$
\begin{aligned}
& - a_4^1 \int_{-\infty}^{\infty} (uu^*_{xxxx} + u^* u_{xxxx})\, dx - ia_5^1 \int_{-\infty}^{\infty} (uu^*_{xxxxx} - u^* u_{xxxxx})\, dx \\
& - a_6^1 \int_{-\infty}^{\infty} (uu^*_{xxxxxx} - u^* u_{xxxxxx})\, dx + ia_1^2 \int_{-\infty}^{\infty} (vv^*_x - v^* v_x)\, dx - a_2^2 \int_{-\infty}^{\infty} (vv^*_{xx} + v^* v_{xx})\, dx \\
& - ia_3^2 \int_{-\infty}^{\infty} (vv^*_{xxx} - v^* v_{xxx})\, dx - a_4^2 \int_{-\infty}^{\infty} (vv^*_{xxxx} + v^* v_{xxxx})\, dx \\
& - ia_5^2 \int_{-\infty}^{\infty} (vv^*_{xxxxx} - v^* v_{xxxxx})\, dx - a_6^2 \int_{-\infty}^{\infty} (vv^*_{xxxxxx} - v^* v_{xxxxxx})\, dx \\
& - b_1^1 \int_{-\infty}^{\infty} |u|^4\, dx - b_1^2 \int_{-\infty}^{\infty} |v|^4\, dx - b_2^2 \int_{-\infty}^{\infty} |u|^2 |v|^2\, dx \\
= & \frac{4A_1^2}{3465B} \left[3 \left\{ 3311 a_1^1 \kappa_1 + a_2^1 \left(3311\kappa_1^2 + 9801B^2 \right) + a_3^1 \kappa_1 \left(3311\kappa_1^2 + 29403B^2 \right) \right.\right. \\
& - a_4^1 \left(3311\kappa_1^4 + 58806\kappa_1^2 B^2 + 3729B^4 \right) - a_5^1 \kappa_1 \left(3311\kappa_1^5 + 98010\kappa_1^2 B^2 - 18645B^4 \right) \\
& \left.\left. + a_6^1 \left(3311\kappa_1^6 + 147015\kappa_1^4 B^2 + 5935\kappa_1^2 B^4 + 40861B^6 \right) \right\} \right] \\
& + \frac{4A_2^2}{3465B} \left[3 \left\{ 3311 a_1^2 \kappa_2 + a_2^2 \left(3311\kappa_2^2 + 9801B^2 \right) + a_3^2 \kappa_2 \left(3311\kappa_2^2 + 29403B^2 \right) \right.\right. \\
& - a_4^2 \left(3311\kappa_2^4 + 58806\kappa_2^2 B^2 + 3729B^4 \right) - a_5^2 \kappa_2 \left(3311\kappa_2^5 + 98010\kappa_2^2 B^2 - 18645B^4 \right) \\
& \left.\left. + a_6^2 \left(3311\kappa_2^6 + 147015\kappa_2^4 B^2 + 5935\kappa_2^2 B^4 + 40861B^6 \right) \right\} \right] \\
& - \frac{5804}{385B} \left(b_1^1 A_1^4 + b_1^2 A_2^4 + b_2^2 A_1^2 A_2^2 \right),
\end{aligned}
\tag{3.11}
$$

which represent the total power, linear momentum and the Hamiltonian, respectively. In (3.9)–(3.11), the parameters A_j for $j = 1, 2$ are the amplitudes of the solitons along u and v components and B is their inverse width.

4. POLYNOMIAL LAW

The NLSE with polynomial (AKA cubic-quintic-septic) law for polarization-preserving fibers is cast as

$$
iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + \left(b_1 |q|^2 + b_2 |q|^4 + b_3 |q|^6 \right) q = 0.
\tag{4.1}
$$

The constants b_1, b_2, b_3 come with cubic, quintic and septic effects, respectively. For birefringent fibers, this NLSE is split. The two components are now written as

$$
\begin{aligned}
& iu_t + ia_1^1 u_x + a_2^1 u_{xx} + ia_3^1 u_{xxx} + a_4^1 u_{xxxx} + ia_5^1 u_{xxxxx} + a_6^1 u_{xxxxxx} + \left(b_{11}^1 |u|^2 + b_{12}^1 |v|^2 \right) u \\
& + \left(b_{11}^2 |u|^4 + b_{12}^2 |u|^2 |v|^2 + b_{13}^2 |v|^4 \right) u + \left(b_{11}^3 |u|^6 + b_{12}^3 |u|^4 |v|^2 + b_{13}^3 |u|^2 |v|^4 + b_{14}^3 |v|^6 \right) u = 0,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
& iv_t + ia_1^2 v_x + a_2^2 v_{xx} + ia_3^2 v_{xxx} + a_4^2 v_{xxxx} + ia_5^2 v_{xxxxx} + a_6^2 v_{xxxxxx} + \left(b_{21}^1 |v|^2 + b_{22}^1 |u|^2 \right) v \\
& + \left(b_{21}^2 |v|^4 + b_{22}^2 |v|^2 |u|^2 + b_{23}^2 |u|^4 \right) v + \left(b_{21}^3 |v|^6 + b_{22}^3 |v|^4 |u|^2 + b_{23}^3 |v|^2 |u|^4 + b_{24}^3 |u|^6 \right) v = 0.
\end{aligned}
\tag{4.3}
$$

The constants b_{j1}^1, b_{j1}^2, b_{j1}^3 and b_{j2}^1, b_{j2}^2, b_{j3}^2, b_{j2}^3, b_{j3}^3, b_{j4}^3 are effects from SPM and XPM, respectively, for $j = 1, 2$. In this case, 4WM effect is discarded.

Bright solitons for (4.2) and (4.3) are [17]

$$
\begin{aligned}
u(x,t) = \pm & \sqrt{\frac{2\begin{pmatrix} 15{\kappa_1}^4 a_6^1 - 10 a_5^1 {\kappa_1}^3 + 300 {\kappa_1}^2 a_6^1 - 6{\kappa_1}^2 a_4^1 \\ -100 a_5^1 \kappa_1 + 3 a_3^1 \kappa_1 + a_2^1 + 616 a_6^1 - 20 a_4^1 \end{pmatrix}}{b_{12}^1 + b_{11}^1}} \\
& \times \operatorname{sech}\left[x - \begin{pmatrix} 5 a_5^1 {\kappa_1}^4 - 6 a_6^1 {\kappa_1}^5 - 2 a_2^1 \kappa_1 \\ -3 a_3^1 {\kappa_1}^2 + 4 a_4^1 {\kappa_1}^3 + a_1^1 \end{pmatrix} t\right] e^{i(-\kappa_1 x + w_1 t + \zeta_1)},
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
v(x,t) = \pm & \sqrt{\frac{2\begin{pmatrix} 15{\kappa_2}^4 a_6^2 - 10 a_5^2 {\kappa_2}^3 + 300 {\kappa_2}^2 a_6^2 - 6{\kappa_2}^2 a_4^2 \\ -100 a_2^2 \kappa_2 + 3 a_3^2 \kappa_2 + a_2^2 + 616 a_6^2 - 20 a_4^2 \end{pmatrix}}{b_{22}^1 + b_{21}^1}} \\
& \times \operatorname{sech}\left[x - \begin{pmatrix} 5 a_5^2 {\kappa_2}^4 - 6 a_6^2 {\kappa_2}^5 - 2 a_2^2 \kappa_2 \\ -3 a_3^2 {\kappa_2}^2 + 4 a_4^2 {\kappa_2}^3 + a_1^2 \end{pmatrix} t\right] e^{i(-\kappa_2 x + w_2 t + \zeta_2)}.
\end{aligned} \tag{4.5}
$$

4.1. Conservation Laws

In general, the conserved density of the complex system is

$$\Phi^t = |u|^2 + |v|^2.$$

For $b_{12}^1 = b_{22}^1$, $b_{13}^3 = b_{23}^3$, $b_{12}^2 = 2b_{23}^2$, $b_{13}^2 = \frac{1}{2}b_{22}^2$, $b_{12}^3 = 3b_{24}^3$ and $b_{14}^3 = \frac{1}{3}b_{22}^3$, the conserved momentum density is

$$\Phi^t = \mathcal{I}(u^* u_x) + \mathcal{I}(v^* v_x)$$

and the conserved Hamiltonian density, in terms of p, w, q and z is

$$
\begin{aligned}
T^t = {} & \frac{1}{4}b_{23}^3\, p^4 z^4 + \frac{1}{6}b_{21}^2\, q^6 + \frac{1}{8}b_{21}^3\, q^8 + \frac{1}{4}b_{11}^1\, p^4 + \frac{1}{4}b_{21}^1\, q^4 + \frac{1}{6}\, b_{11}^2\, p^6 + \frac{1}{8}b_{11}^3\, p^8 + z^2 b_{23}^3\, p^2 w^2 q^2 \\
& + \frac{1}{6}b_{22}^3\, w^2 z^6 + \frac{1}{4}b_{23}^3\, w^4 z^4 + \frac{1}{2}z^2 b_{24}^3\, w^6 + \frac{1}{4}b_{22}^2\, w^2 z^4 + \frac{1}{8}b_{11}^3\, w^8 + \frac{1}{2}w^2 b_{11}^2\, p^4 + \frac{1}{4}w^2 b_{22}^2\, q^4 \\
& + \frac{1}{2}w^2 b_{11}^3\, p^6 + \frac{1}{2}b_{11}^2\, p^2 w^4 + \frac{1}{6}w^2 b_{22}^3\, q^6 + 1/2\, b_{23}^2\, w^4 q^2 + \frac{1}{2}q^2 b_{21}^1\, p^2 + \frac{1}{2}\, q^2 b_{23}^2\, p^4 + \frac{1}{2}q^2 b_{24}^3\, p^6 \\
& + \frac{1}{4}b_{22}^2\, p^2 q^4 + \frac{1}{2}z^2 b_{21}^2\, q^4 + \frac{1}{2}z^2 b_{23}^2\, p^4 + \frac{1}{2}z^2 b_{24}^3\, p^6 + \frac{1}{2}\, z^2 b_{21}^3\, q^6 + \frac{1}{2}b_{21}^2\, q^2 z^4 + \frac{1}{4}b_{22}^2\, p^2 z^4 \\
& + \frac{1}{2}b_{21}^3\, q^2 z^6 + \frac{3}{4}\, b_{21}^3\, q^4 z^4 + \frac{1}{6}b_{22}^3\, p^2 z^6 + \frac{1}{2}b_{24}^3\, w^6 q^2 + \frac{1}{2}b_{11}^3\, p^2 w^6 + \frac{1}{4}b_{23}^3\, w^4 q^4 + \frac{1}{2}z a_2^1\, q_x \\
& + \frac{1}{2}p a_6^1\, p_{xxxxxx} - \frac{1}{2}p a_5^1\, w_{xxxxx} + \frac{1}{2}p a_4^1\, p_{xxxx} + \frac{1}{2}w a_6^1\, w_{xxxxxx} + \frac{1}{2}w a_4^1\, w_{xxxx} + \frac{1}{2}w a_5^1\, p_{xxxxx} \\
& + \frac{1}{2}w a_2^1\, w_{xx} + \frac{1}{2}w a_3^1\, p_{xxx} + \frac{1}{2}w a_1^1\, p_x + \frac{1}{2}w^2 b_{11}^1\, p^2 + \frac{1}{2}w^2 b_{21}^1\, q^2 + \frac{3}{4}\, b_{11}^3\, p^4 w^4 + \frac{1}{6}b_{22}^3\, p^2 q^6 \\
& + \frac{1}{4}\, b_{23}^3\, p^4 q^4 - \frac{1}{2}q a_2^1\, z_x - \frac{1}{2}p a_3^1\, w_{xxx} - \frac{1}{2}p a_1^1\, w_x + \frac{1}{2}p a_2^1\, p_{xx} + \frac{1}{2}q a_6^2\, q_{xxxxxx} + \frac{1}{2}q a_4^2\, q_{xxxx} \\
& - \frac{1}{2}q a_5^2\, z_{xxxxx} + \frac{1}{2}z^2 b_{21}^1\, q^2 + \frac{1}{2}q a_2^2\, q_{xx} - \frac{1}{2}q a_3^2\, z_{xxx} + \frac{1}{2}z^2 b_{21}^1\, p^2 + \frac{1}{2}z a_6^2\, z_{xxxxxx} + \frac{1}{2}z a_5^2\, q_{xxxxx} \\
& + \frac{1}{2}z a_4^2\, z_{xxxx} + \frac{1}{2}z a_3^2\, q_{xxx} + \frac{1}{2}z a_2^2\, z_{xx} + \frac{1}{2}z^2 b_{22}^2\, p^2 q^2 + \frac{1}{2}b_{22}^3\, p^2 q^2 z^4 + \frac{1}{2}b_{22}^3\, w^2 q^2 z^4 \\
& + \frac{1}{2}b_{23}^3\, p^2 w^2 z^4 + \frac{1}{2}z^2 b_{22}^2\, w^2 q^2 + z^2 b_{23}^2\, p^2 w^2 + \frac{1}{2}z^2 b_{22}^3\, p^2 q^4 + \frac{1}{2}z^2 b_{22}^3\, w^2 q^4 + \frac{1}{2}z^2 b_{23}^3\, p^4 q^2
\end{aligned}
$$

$$+\frac{1}{2}z^2b_{23}^3\,w^4q^2+\frac{3}{2}z^2b_{24}^3\,p^4w^2+\frac{3}{2}z^2b_{24}^3\,p^2w^4+w^2b_{23}^2\,p^2q^2+\frac{3}{2}w^2b_{24}^3\,p^4q^2+\frac{1}{2}w^2b_{23}^3\,p^2q^4$$
$$+\frac{3}{2}b_{24}^3\,p^2w^4q^2+\frac{1}{2}\,z^2b_{21}^1\,w^2+\frac{1}{2}z^2b_{23}^2\,w^4+\frac{1}{6}b_{11}^2\,w^6+\frac{1}{4}b_{21}^1\,z^4+\frac{1}{6}b_{21}^2\,z^6+\frac{1}{8}b_{21}^3\,z^8+\frac{1}{4}b_{11}^1\,w^4, \tag{4.6}$$

so that

$$\Phi^t=-\frac{1}{2}\big[a_1^1\mathcal{I}(u^*u_x)+a_3^1\mathcal{I}(u^*u_{xxx})+a_5^1\mathcal{I}(u^*u_{xxxxx})\big]+\frac{1}{4}\big[a_2^1\mathcal{R}(uu_{xx}^*)+a_4^1\mathcal{R}(uu_{xxxx}^*)$$
$$+a_6^1\mathcal{R}(uu_{xxxxxx}^*)\big]-\frac{1}{2}[a_2^1\mathcal{I}(v^*v_x)+a_3^2\mathcal{I}(v^*v_{xxx})+a_5^2\mathcal{I}(v^*v_{xxxxx})]+\frac{1}{4}\big[a_2^2\mathcal{R}(vv_{xx}^*)$$
$$+a_4^2\mathcal{R}(vv_{xxxx}^*)+a_6^2\mathcal{R}(vv_{xxxxxx}^*)\big]+\frac{1}{2}b_{12}^1|u|^2|v|^2+\frac{1}{4}b_{11}^1|u|^4+\frac{1}{6}b_{11}^2|u|^6+\frac{1}{8}b_{11}^3|u|^8$$
$$+b_{13}^3\frac{1}{4}|u|^4|v|^4+\frac{1}{6}b_{21}^2|v|^6+\frac{1}{8}b_{21}^3|v|^8+\frac{1}{4}b_{12}^1|u|^4|v|^2+\frac{1}{4}b_{22}^2|v|^4|u|^2+\frac{1}{2}b_{22}^3|u|^2|v|^6$$
$$+\frac{1}{2}b_{24}^3|u|^6|v|^2+\frac{1}{4}b_{23}^3|u|^4|v|^4.$$

Thus, the conserved quantities are

$$P=\int_{-\infty}^{\infty}\Big(|u|^2+|v|^2\Big)\,dx=\frac{2}{B}\left(A_1^2+A_2^2\right), \tag{4.7}$$

$$M=2i\int_{-\infty}^{\infty}\{(u^*u_x-uu_x^*)+(v^*v_x-vv_x^*)\}\,dx=\frac{2}{B}\left(\kappa_1A_1^2+\kappa_2A_2^2\right) \tag{4.8}$$

and

$$H=12\Bigg[ia_1^1\int_{-\infty}^{\infty}(uu_x^*-u^*u_x)\,dx-a_2^1\int_{-\infty}^{\infty}(uu_{xx}^*+u^*u_{xx})\,dx-ia_3^1\int_{-\infty}^{\infty}(uu_{xxx}^*-u^*u_{xxx})\,dx$$
$$-\,a_4^1\int_{-\infty}^{\infty}(uu_{xxxx}^*+u^*u_{xxxx})\,dx-ia_5^1\int_{-\infty}^{\infty}(uu_{xxxxx}^*-u^*u_{xxxxx})\,dx$$
$$-a_6^1\int_{-\infty}^{\infty}(uu_{xxxxxx}^*-u^*u_{xxxxxx})\,dx+ia_1^2\int_{-\infty}^{\infty}(vv_x^*-v^*v_x)\,dx-a_2^2\int_{-\infty}^{\infty}(vv_{xx}^*+v^*v_{xx})\,dx$$
$$-ia_3^2\int_{-\infty}^{\infty}(vv_{xxx}^*-v^*v_{xxx})\,dx-a_4^2\int_{-\infty}^{\infty}(vv_{xxxx}^*+v^*v_{xxxx})\,dx-ia_5^2\int_{-\infty}^{\infty}(vv_{xxxxx}^*-v^*v_{xxxxx})\,dx$$
$$-a_6^2\int_{-\infty}^{\infty}(vv_{xxxxxx}^*-v^*v_{xxxxxx})\,dx\Bigg]+6b_{11}^1\int_{-\infty}^{\infty}|u|^4\,dx+4b_{11}^2\int_{-\infty}^{\infty}|u|^6\,dx+3b_{11}^3\int_{-\infty}^{\infty}|u|^8\,dx$$
$$+\,6b_{21}^1\int_{-\infty}^{\infty}|v|^4\,dx+4b_{21}^2\int_{-\infty}^{\infty}|v|^6\,dx+3b_{21}^3\int_{-\infty}^{\infty}|v|^8\,dx-12b_{12}^1\int_{-\infty}^{\infty}|u|^2\,|v|^2\,dx$$
$$-\,6\left(b_{13}^3+b_{23}^3\right)\int_{-\infty}^{\infty}|u|^4\,|v|^4\,dx-6b_{22}^2\int_{-\infty}^{\infty}|u|^2\,|v|^4\,dx-6b_{12}^1\int_{-\infty}^{\infty}|u|^4\,|v|^2\,dx$$
$$-\,12b_{22}^3\int_{-\infty}^{\infty}|u|^2\,|v|^6\,dx-12b_{24}^3\int_{-\infty}^{\infty}|u|^6\,|v|^2\,dx$$
$$=\frac{48A_1^2}{105B}\left[\left\{105a_1^1\kappa_1+35a_2^1\left(3\kappa_1^2+B^2\right)+105a_3^1\kappa_1\left(\kappa_1^2+B^2\right)-7a_4^1\left(15\kappa_1^4-30\kappa_1^2B^2+7B^4\right)\right.\right.$$
$$\left.\left.-\;35a_5^1\kappa_1\left(3\kappa_1^4+10\kappa_1^2B^2+7B^4\right)+5a_6^1\left(21\kappa_1^6+105\kappa_1^4B^2+147\kappa_1^2B^4+31B^6\right)\right\}\right]$$
$$+\,\frac{48A_2^2}{105B}\left[\left\{105a_1^1\kappa_2+35a_2^1\left(3\kappa_2^2+B^2\right)+105a_3^1\kappa_2\left(\kappa_2^2+B^2\right)-7a_4^1\left(15\kappa_2^4-30\kappa_2^2B^2+7B^4\right)\right.\right.$$
$$\left.\left.-\;35a_5^1\kappa_2\left(3\kappa_2^4+10\kappa_2^2B^2+7B^4\right)+5a_6^1\left(21\kappa_2^6+105\kappa_2^4B^2+147\kappa_2^2B^4+31B^6\right)\right\}\right]$$
$$-\,\frac{1}{105B}\left[8A_1^4\left(105b_{11}^1+56b_{11}^2A_1^2+36b_{11}^3A_1^4\right)+8A_2^4\left(105b_{21}^1+56b_{21}^2A_2^2+36b_{21}^3A_2^4\right)\right.$$
$$\left.+\,48A_1^2A_2^2\left\{35b_{12}^1+12\left(b_{13}^3+b_{23}^3\right)A_1^2A_2^2\right\}+672A_1^2A_2^2\left(b_{22}^2A_2^2+b_{12}^1A_1^2\right)+1152A_1^2A_2^2\left(b_{22}^3A_2^4+b_{24}^3A_1^4\right)\right]. \tag{4.9}$$

5. NONLOCAL LAW

The NLSE in polarization-preserving fibers with the nonlocal law is

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + b\left(|q|^2\right)_{xx} q = 0. \tag{5.1}$$

The constant b accounts for nonlocal nonlinearity. For birefringent fibers, the split components are

$$iu_t + ia_1^1 u_x + a_2^1 u_{xx} + ia_3^1 u_{xxx} + a_4^1 u_{xxxx} + ia_5^1 u_{xxxxx} + a_6^1 u_{xxxxxx} + \left\{b_1^1\left(|u|^2\right)_{xx} + b_2^1\left(|v|^2\right)_{xx}\right\} u = 0, \tag{5.2}$$

$$iv_t + ia_1^2 v_x + a_2^2 v_{xx} + ia_3^2 v_{xxx} + a_4^2 v_{xxxx} + ia_5^2 v_{xxxxx} + a_6^2 v_{xxxxxx} + \left\{b_1^2\left(|v|^2\right)_{xx} + b_2^2\left(|u|^2\right)_{xx}\right\} v = 0. \tag{5.3}$$

The constants b_j^l for $l = 1, 2$ represent SPM coefficients for $j = 1$ and XPM coefficients for $j = 2$.

Bright solitons to (5.2) and (5.3) are [17]

$$u(x,t) = \left\{\pm\sqrt{\frac{252\, a_6^1 \delta^2}{b_2^1 + b_1^1}} \pm \sqrt{\frac{252\, a_6^1}{b_2^1 + b_1^1}} \operatorname{sech}^2\left[x - \begin{pmatrix} 5a_5^1 {\kappa_1}^4 - 6a_6^1 {\kappa_1}^5 - 2a_2^1 \kappa_1 \\ -3a_3^1 {\kappa_1}^2 + 4a_4^1 {\kappa_1}^3 + a_1^1 \end{pmatrix} t\right]\right\} \times e^{i(-\kappa_1 x + w_1 t + \zeta_1)}, \tag{5.4}$$

$$v(x,t) = \left\{\pm\sqrt{\frac{252\, a_6^2 \delta^2}{b_2^2 + b_1^2}} \pm \sqrt{\frac{252\, a_6^2}{b_2^2 + b_1^2}} \operatorname{sech}^2\left[x - \begin{pmatrix} 5a_5^2 {\kappa_2}^4 - 6a_6^2 {\kappa_2}^5 - 2a_2^2 \kappa_2 \\ -3a_3^2 {\kappa_2}^2 + 4a_4^2 {\kappa_2}^3 + a_1^2 \end{pmatrix} t\right]\right\} \times e^{i(-\kappa_2 x + w_2 t + \zeta_2)}. \tag{5.5}$$

5.1. Conservation Laws

In the system above, we let $u = p + iw$ and $v = q + iz$, so that the system splits into a system of four PDEs whose conserved flows (T^t, T^x) are constructed using the multiplier approach. It turns out that for arbitrary values of the parameters we have a single multiplier $Q = (-p, -q, w, z)$ giving rise to the conserved density

$$T^t = -\frac{1}{2}(p^2 + w^2 + q^2 + z^2), \tag{5.6}$$

so that a corresponding conserved density of the complex system is

$$\Phi^t = |u|^2 + |v|^2.$$

If $b_2^1 = b_2^2$, then the momentum and the Hamiltonian are conserved, giving rise to the following conserved densities T^t:

Linear momentum:

$$T^t = -\left[-\frac{1}{2} q_x z - \frac{1}{2} p_x w + \frac{1}{2} z_x q + \frac{1}{2} w_x p\right] \tag{5.7}$$

and the momentum density is

$$\Phi^t = \mathcal{I}(u^* u_x) + \mathcal{I}(v^* v_x).$$

Hamiltonian:

$$
\begin{aligned}
T^t = {} & \frac{1}{2} w a_5^1\, p_{xxxxx} - \frac{1}{2} q a_2^1\, z_x + \frac{1}{2} z a_4^2\, z_{xxxx} + \frac{1}{2} w^2 b_1^1\, {p_x}^2 + \frac{1}{2} w^2 b_2^2\, {z_x}^2 + \frac{1}{2} w a_1^1\, p_x \\
& + \frac{1}{2} p^2 b_2^2\, {q_x}^2 + \frac{1}{2} q^2 b_1^2\, {z_x}^2 + \frac{1}{2} q a_6^2\, q_{xxxxxx} + \frac{1}{2} z a_2^1\, q_x + \frac{1}{2} p a_6^1\, p_{xxxxxx} + \frac{1}{2} p^2 b_2^2\, {z_x}^2 \\
& + \frac{1}{2} q^2 b_2^2\, p p_{xx} + \frac{1}{2} w^2 b_2^2\, z z_{xx} + \frac{1}{2} w^2 b_2^2\, q q_{xx} + \frac{1}{2} w^2 b_1^1\, p p_{xx} + \frac{1}{2} q^2 b_2^2\, w w_{xx} \\
& + \frac{1}{2} p^2 b_1^1\, w w_{xx} + \frac{1}{2} q^2 b_1^2\, z z_{xx} + \frac{1}{2} p^2 b_2^2\, q q_{xx} + \frac{1}{2} z^2 b_2^2\, p p_{xx} + \frac{1}{2} z^2 b_2^2\, w w_{xx} \\
& + \frac{1}{2} z^2 b_1^2\, q q_{xx} + \frac{1}{2} p^2 b_2^2\, z z_{xx} + \frac{1}{2} q a_4^2\, q_{xxxx} + \frac{1}{2} w a_3^1\, p_{xxx} + \frac{1}{2} q a_2^2\, q_{xx} + \frac{1}{2} w a_6^1\, w_{xxxxxx} \\
& - \frac{1}{2} p a_1^1\, w_x + \frac{1}{2} p a_2^1\, p_{xx} + \frac{1}{2} b_1^2\, q^3 q_{xx} + \frac{1}{2} q^2 b_2^2\, {w_x}^2 + \frac{1}{2} b_1^1\, p^3 p_{xx} - \frac{1}{2} p a_3^1\, w_{xxx} \\
& + \frac{1}{2} q^2 b_2^2\, {p_x}^2 + \frac{1}{2} w a_4^1\, w_{xxxx} - \frac{1}{2} q a_3^2\, z_{xxx} + \frac{1}{2} w^2 b_2^2\, {q_x}^2 + \frac{1}{2} q^2 b_1^2\, {q_x}^2 + \frac{1}{2} p^2 b_1^1\, {p_x}^2 \\
& + \frac{1}{2} w^2 b_1^1\, {w_x}^2 + \frac{1}{2} b_1^1\, w^3 w_{xx} + \frac{1}{2} w a_2^1\, w_{xx} + \frac{1}{2} p^2 b_1^1\, {w_x}^2 - \frac{1}{2} q a_5^2\, z_{xxxxx} - \frac{1}{2} p a_5^1\, w_{xxxxx} \\
& + \frac{1}{2} p a_4^1\, p_{xxxx} + \frac{1}{2} z a_6^2\, z_{xxxxxx} + \frac{1}{2} z a_3^2\, q_{xxx} + \frac{1}{2}\, b_1^2\, z^3 z_{xx} + \frac{1}{2} z^2 b_1^2\, {z_x}^2 \\
& + \frac{1}{2} z^2 b_2^2\, {w_x}^2 + \frac{1}{2} z^2 b_1^2\, {q_x}^2 + \frac{1}{2} z^2 b_2^2\, {p_x}^2 + \frac{1}{2} z a_5^2\, q_{xxxxx} + \frac{1}{2} z a_2^2\, z_{xx},
\end{aligned}
\tag{5.8}
$$

so that

$$
\begin{aligned}
\Phi^t = {} & \frac{1}{2}\big[-a_1^1\mathcal{I}(u^*u_x) + a_2^1\mathcal{R}(uu_{xx}^*) - a_3^1\mathcal{I}(u^*u_{xxx}) + a_4^1\mathcal{R}(uu_{xxxx}^*) - a_5^1\mathcal{I}(u^*u_{xxxxx}) \\
& + a_6^1\mathcal{R}(uu_{xxxxxx}^*) - a_1^2\mathcal{I}(v^*v_x) + a_2^2\mathcal{R}(vv_{xx}^*) - a_3^2\mathcal{I}(v^*v_{xxx}) + a_4^2\mathcal{R}(vv_{xxxx}^*) \\
& - a_5^2\mathcal{I}(v^*v_{xxxxx}) + a_6^2\mathcal{R}(vv_{xxxxxx}^*) + b_1^1|u|^2|u_x|^2 + b_1^2|v|^2|v_x|^2 + b_2^2(|u|^2|v_x|^2 + |v|^2|u_x|^2) \\
& + b_2^2(|u|^2\mathcal{R}(vv_{xx}^*) + |v|^2\mathcal{R}(uu_{xx}^*)) + b_1^1|u|^2\mathcal{R}(uu_{xx}^*) + b_1^2|v|^2\mathcal{R}(vv_{xx})\big].
\end{aligned}
$$

In this case, the conserved quantities are

$$
P = \int_{-\infty}^{\infty} \Big(|u|^2 + |v|^2\Big)\, dx = \frac{4}{3B}\left(A_1^2 + A_2^2\right), \tag{5.9}
$$

$$
M = 2i \int_{-\infty}^{\infty} \{(u^*u_x - uu_x^*) + (v^*v_x - vv_x^*)\}\, dx = \frac{4}{3B}\left(\kappa_1 A_1^2 + \kappa_2 A_2^2\right) \tag{5.10}
$$

and

$$
\begin{aligned}
H = {} & ia_1^1 \int_{-\infty}^{\infty} (uu_x^* - u^*u_x)\, dx - a_2^1 \int_{-\infty}^{\infty} (uu_{xx}^* + u^*u_{xx})\, dx - ia_3^1 \int_{-\infty}^{\infty} (uu_{xxx}^* - u^*u_{xxx})\, dx \\
& - a_4^1 \int_{-\infty}^{\infty} (uu_{xxxx}^* + u^*u_{xxxx})\, dx - ia_5^1 \int_{-\infty}^{\infty} (uu_{xxxxx}^* - u^*u_{xxxxx})\, dx \\
& - a_6^1 \int_{-\infty}^{\infty} (uu_{xxxxxx}^* - u^*u_{xxxxxx})\, dx + ia_1^2 \int_{-\infty}^{\infty} (vv_x^* - v^*v_x)\, dx - a_2^2 \int_{-\infty}^{\infty} (vv_{xx}^* + v^*v_{xx})\, dx \\
& - ia_3^2 \int_{-\infty}^{\infty} (vv_{xxx}^* - v^*v_{xxx})\, dx - a_4^2 \int_{-\infty}^{\infty} (vv_{xxxx}^* + v^*v_{xxxx})\, dx - ia_5^2 \int_{-\infty}^{\infty} (vv_{xxxxx}^* - v^*v_{xxxxx})\, dx \\
& - a_6^2 \int_{-\infty}^{\infty} (vv_{xxxxxx}^* - v^*v_{xxxxxx})\, dx - b_1^1 \int_{-\infty}^{\infty} |u|^2 \left\{|u_x|^2 + (uu_{xx}^* + u^*u_{xx})\right\} dx \\
& - b_1^2 \int_{-\infty}^{\infty} |v|^2 \left\{|v_x|^2 + (vv_{xx}^* + v^*v_{xx})\right\} dx - b_2^2 \left(|u|^2\,|v_x|^2 + |v|^2\,|u_x|^2\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4A_1^2}{315B}\left[\left\{210a_1^1\kappa_1 + 42a_2^1\left(5\kappa_1^2+4B^2\right) + 42a_3^1\kappa_1\left(5\kappa_1^2+12B^2\right) - 6a_4^1\left(35\kappa_1^4+168\kappa_1^2B^2+80B^4\right)\right.\right.\\
&\quad\left.\left. -\ 30a_5^1\kappa_1\left(7\kappa_1^4+56\kappa_1^2B^2+80B^4\right) + 6a_6^1\left(35\kappa_1^6+420\kappa_1^4B^2+1200\kappa_1^2B^4+448B^6\right)\right\}\right]\\
&\quad + \frac{4A_2^2}{315B}\left[\left\{210a_1^2\kappa_2 + 42a_2^2\left(5\kappa_2^2+4B^2\right) + 42a_3^2\kappa_2\left(5\kappa_2^2+12B^2\right) - 6a_4^2\left(35\kappa_2^4+168\kappa_2^2B^2+80B^4\right)\right.\right.\\
&\quad\left.\left. -\ 30a_5^2\kappa_2\left(7\kappa_2^4+56\kappa_2^2B^2+80B^4\right) + 6a_6^2\left(35\kappa_2^6+420\kappa_2^4B^2+1200\kappa_2^2B^4+448B^6\right)\right\}\right]\\
&\quad - \frac{32}{315B}\left\{b_1^1A_1^4\left(9\kappa_1^2+20B^2\right) + b_1^2A_2^4\left(9\kappa_2^2+20B^2\right) + 3b_2^2A_1^2A_2^2\left(3\kappa_1^2+3\kappa_2^2-4B^2\right)\right\}.
\end{aligned}
$$

6. CONCLUSIONS

This paper secures conservation laws for HD solitons in birefringent fibers that have three forms of nonlinear refractive index. They are Kerr law, polynomial law and nonlocal law. The three conservation laws that emerge with the application of multiplier scheme are power, linear momentum and Hamiltonian. The conserved quantities are computed from the bright 1-soliton solutions that have been reported in an earlier work [17]. Later, this work needs to be extended further. The next avenue to move to is to address the model with DWDM topology. Thus, HD solitons will be addressed with DWDM topology and finally their conservation laws will be studied. Hence, there is a lot up on the table for grabs. Those results are currently awaited.

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CONFLICT OF INTEREST

The authors also declare that there is no conflict of interest.

REFERENCES

1. Biswas, A., Vega-Guzman, J., Mahmood, M. F., Khan, S., Ekici, M., Zhou, Q., Moshokoa, S. P., and Belic, M. R., Highly Dispersive Optical Solitons with Undetermined Coefficients, *Optik,* 2019, vol. 182, pp. 890–896.
2. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Kerr Law Nonlinearity by F-Expansion, *Optik,* 2019, vol. 181, pp. 1028–1038.
3. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Quadratic-Cubic Law by F-Expansion, *Optik,* 2019, vol. 182, pp. 930–943.
4. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Kerr Law Nonlinearity by Extended Jacobi's Elliptic Function Expansion, *Optik,* 2019, vol. 183, pp. 395–400.
5. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Non-Local Nonlinearity by Extended Jacobi's Elliptic Function Expansion, *Optik,* 2019, vol. 184, pp. 277–286.
6. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Non-Local Nonlinearity by F-Expansion, *Optik,* 2019, vol. 186, pp. 288–292.
7. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Cubic-Quintic-Septic Law by F-Expansion, *Optik,* 2019, vol. 182, pp. 897–906.
8. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Cubic-Quintic-Septic Law by exp-Expansion, *Optik,* 2019, vol. 186, pp. 321–325.
9. Biswas, A., Ekici, M., Sonmezoglu, A., and Belic, M. R., Highly Dispersive Optical Solitons with Cubic-Quintic-Septic Law by Extended Jacobi's Elliptic Function Expansion, *Optik,* 2019, vol. 183, pp. 571–578.
10. Biswas, A., Kara, A. H., Alshomrani, A. S., Ekici, M., Zhou, Q., and Belic, M. R., Conservation Laws for Highly Dispersive Optical Solitons, *Optik,* 2019, vol. 199, Art. 163283.
11. Kohl, R. W., Biswas, A., Ekici, M., Zhou, Q., Khan, S., Alshomrani, A. S., and Belic, M. R., Highly Dispersive Optical Soliton Perturbation with Kerr Law Nonlinearity by Semi-Inverse Variational Principle, *Optik,* 2019, vol. 199, Art. 163226.
12. Kohl, R. W., Biswas, A., Ekici, M., Zhou, Q., Khan, S., Alshomrani, A. S., and Belic, M. R., Highly Dispersive Optical Soliton Perturbation with Cubic-Quintic-Septic Refractive Index by Semi-Inverse Variational Principle, *Optik,* 2019, vol. 199, Art. 163322.

13. Kudryashov, N. A., Highly Dispersive Solitary Wave Solutions of Perturbed Nonlinear Schrödinger Equations, *Appl. Math. Comput.,* 2020, vol. 371, 124972, 11 pp.
14. Kudryashov, N. A., Solitary Wave Solutions of Hierarchy with Non-Local Nonlinearity, *Appl. Math. Lett.,* 2020, vol. 103, 106155, 5 pp.
15. Kudryashov, N. A., Method for Finding Highly Dispersive Optical Solitons of Nonlinear Differential Equations, *Optik,* 2020 (in press).
16. Kudryashov, N. A., Highly Dispersive Optical Solitons of the Generalized Nonlinear Eighth-Order Schrödinger Equation, *Optik,* 2020, vol. 206, Art. 164335.
17. Yildirim, Y., Biswas, A., Ekici, M., Khan, S., Moraru, L., Alzahrani, A. K., and Belic, M. R., Highly Dispersive Optical Solitons in Birefringent Fibers with Four Forms of Nonlinear Refractive Index by Three Prolific Integration Schemes, *submitted for publication* (2019).
18. Rehman, H. U., Ullah, N., and Imran, M. A., Highly Dispersive Optical Solitons Using Kudryashov's Method, *Optik,* 2019, vol. 199, Art. 163349.
19. Triki, H. and Kruglov, V. I., Propagation of Dipole Solitons in Inhomogeneous Highly Dispersive Optical Fiber Media, *submitted for publication* (2019).
20. Anco, S. C. and Bluman, G., Direct Construction Method for Conservation Laws of Partial Differential Equations: 1. Examples of Conservation Law Classifications, *European J. Appl. Math.,* 2002, vol. 13, no. 5, pp. 545–566.
21. Anco, S. C. and Bluman, G., Direct Construction Method for Conservation Laws of Partial Differential Equations: 2. General Treatment, *European J. Appl. Math.,* 2002, vol. 13, no. 5, pp. 567–585.
22. Anderson, I. M. and Duchamp, T. E., Variational Principles for Second-Order Quasi-Linear Scalar Equations, *J. Differential Equations,* 1984, vol. 51, no. 1, pp.,1–47.
23. Anderson, I. M. and Pohjanpelto, J., The Cohomology of Invariant Variational Bicomplexes, *Acta Appl. Math.,* 1995, vol. 41, nos. 1–3, pp. 3–19.
24. Kara, A. H. and Mahomed, F. M., Relationship between Symmetries and Conservation Laws, *Int. J. Theor. Phys.,* 2000, vol. 39, no. 1, pp. 23–40.
25. Poole, D. and Hereman, W., The Homotopy Operator Method for Symbolic Integration by Parts and Inversion of Divergences with Applications, *Appl. Anal.,* 2010, vol. 89, no. 4, pp. 433–455.