Optics

Embedded solitons with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinear susceptibilities

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Abstract. Studied in this work are embedded solitons with quadratic nonlinearity that includes the effect of spatio-temporal dispersion. Two integration schemes yield bright, dark, singular and combo singular soliton solutions from the continuous regime. The existence criteria for these solitons are also included.

Keywords: $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities, embedded solitons.

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1. Introduction

The study of optical solitons in discrete spectrum is quite widespread. There are several results that are reported in its avenue. However, quite less attention has been paid to soliton studies that stem from the continuous regime. There are a few results that have been reported in this context [1–10]. This paper revisits the study of embedded solitons with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinear susceptibilities. There are two efficient integration schemes that are implemented in today's work to recover soliton solutions to the model. These give away to bright, dark, singular and combo singular forms of embedded solitons. The existence criteria for these solitons are also listed. The details of the integration procedures along with the spectrum of soliton solutions are all enlisted in the rest of the paper, but first, the governing model with its physical interpretation is illustrated.

1.1. Governing model

The governing model with the quadratic nonlinearity [1–10] is as follows:

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 q^* r + d_1 |q|^2 q = 0\,, \qquad (1)$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 r + d_2 q^2 + \delta |q|^2 r = 0. \qquad (2)$$

In this model, the independent variables x and t represent the spatial and temporal variables, respectively. The constants a_j provide the chromatic dispersion, while the constants b_j assure the existence of spatio-temporal dispersion (STD). Also, the coefficients c_j provide the existence of group-velocity mismatch because of frequency difference between fundamental harmonics (FH) and second harmonics (SH) fields that are given by the complex valued functions $q(x, t)$, $r(x, t)$, respectively. This model specifically governs embedded solitons that are nonlinear waves and become confined to the continuous spectrum of a nonlinear system. These solitons arise in presence of opposing dispersion and competing nonlinearities at FH and SH.

2. Mathematical analysis

To start off, the basic assumptions are

$$q(x,t) = P_1(\zeta) e^{i\varphi(x,t)}, \qquad (3)$$

$$r(x,t) = P_2(\zeta) e^{2i\varphi(x,t)}, \qquad (4)$$

where

$$\zeta = \eta(x - vt), \qquad (5)$$

and v stands for the speed of the wave. Next, the phase φ is structured as

$$\varphi(x,t)=-\kappa x+\omega t+\theta_0\,, \tag{6}$$

where the frequency, wave number and phase constant are designated as κ, ω and θ_0, respectively. Insert (3) and (4) into (1) and (2). Then, from (1), real and imaginary parts fall out

$$\eta^2(b_1v-a_1)P_1''+\left(\omega+a_1\kappa^2-b_1\omega\kappa\right)P_1-c_1P_1P_2-d_1P_1^3=0\,, \tag{7}$$

$$v=\frac{b_1\omega-2a_1\kappa}{1-b_1\kappa}\,, \tag{8}$$

respectively. And then from (2)

$$\eta^2(b_2v-a_2)P_2''+\left(2\omega+4a_2\kappa^2-4b_2\omega\kappa-c_2\right)P_2- \\ -d_2P_1^2-\delta P_1^2P_2=0\,, \tag{9}$$

$$v=\frac{2b_2\omega-4a_2\kappa}{1-2b_2k}\,, \tag{10}$$

Eqs (7) – (10) reduce to

$$2\eta^2(bv-a)P_1''+\left(\omega+2a\kappa^2-2b\omega\kappa\right)P_1- \\ -c_1P_1P_2-d_1P_1^3=0\,, \tag{11}$$

$$\eta^2(bv-a)P_2''+\left(2\omega+4a\kappa^2-4b\omega\kappa-c_2\right)P_2- \\ -d_2P_1^2-\delta P_1^2P_2=0\,, \tag{12}$$

$$v=\frac{2b\omega-4a\kappa}{1-2bk}\,, \tag{13}$$

as long as

$$a_1=2a,\ a_2=a,\ b_1=2b,\ b_2=b\,. \tag{14}$$

Thus, the governing equations can be written as

$$iq_t+2aq_{xx}+2bq_{xt}+c_1q^*r+d_1|q|^2q=0\,, \tag{15}$$

$$ir_t+ar_{xx}+br_{xt}+c_2r+d_2q^2+\delta|q|^2r=0\,. \tag{16}$$

Eqs (11) and (12) are employed to retrieve solitons for Eqs (15) and (16).

2.1. F-expansion procedure

The solution structure of (11) and (12) is considered to be

$$P_1(\zeta)=\sum_{i=0}^{N}A_iF^i(\zeta), \tag{17}$$

$$P_2(\zeta)=\sum_{i=0}^{N}B_iF^i(\zeta)\,, \tag{18}$$

where A_i and B_i for $1\le i\le N$ are constants that need to be designated, and the number N originates from balancing principle. The function $F(\zeta)$ obeys the form:

$$F'(\zeta)=\sqrt{PF^2(\zeta)+QF(\zeta)+R}\,, \tag{19}$$

where P, Q and R are constants. It is necessary to note that the solutions of (19) are as follows:

$$\begin{aligned}
&F(\zeta)=\mathrm{sn}(\zeta)=\tanh(\zeta),\ P=m^2,\\
&Q=-(1+m^2),\ R=1,\ m\to1,\\
&F(\zeta)=\mathrm{ns}(\zeta)=\coth(\zeta),\ P=1,\\
&Q=-(1+m^2),\ R=m^2,\ m\to1,\\
&F(\zeta)=\mathrm{sc}(\zeta)=\tan(\zeta),\ P=1-m^2,\\
&Q=2-m^2,\ R=1,\ m\to0,\\
&F(\zeta)=\mathrm{cs}(\zeta)=\cot(\zeta),\ P=1,\\
&Q=2-m^2,\ R=1-m^2,\ m\to0,\\
&F(\zeta)=\mathrm{cn}(\zeta)=\mathrm{sech}(\zeta),\ P=-m^2,\\
&Q=2m^2-1,\ R=1-m^2,\ m\to1,\\
&F(\zeta)=\mathrm{ds}(\zeta)=\mathrm{csch}(\zeta),\ P=1,\\
&Q=2m^2-1,\ R=-m^2\left(1-m^2\right),\ m\to1,\\
&F(\zeta)=\mathrm{nc}(\zeta)=\sec(\zeta),\ P=1-m^2,\\
&Q=2m^2-1,\ R=-m^2,\ m\to0,\\
&F(\zeta)=\mathrm{ns}(\zeta)=\csc(\zeta),\ P=1,\\
&Q=-(1+m^2),\ R=m^2,\ m\to0,\\
&F(\zeta)=\mathrm{ns}(\zeta)\pm\mathrm{ds}(\zeta)=\coth(\zeta)\pm\mathrm{csch}(\zeta),\ P=\frac{1}{4},\\
&Q=\frac{m^2-2}{2},\ R=\frac{m^2}{4},\ m\to1,\\
&F(\zeta)=\mathrm{sn}(\zeta)\pm i\,\mathrm{cn}(\zeta)=\tanh(\zeta)\pm i\,\mathrm{sech}(\zeta),\ P=\frac{m^2}{4},\\
&Q=\frac{m^2-2}{2},\ R=\frac{m^2}{4},\ m\to1,\\
&F(\zeta)=\mathrm{ns}(\zeta)\pm\mathrm{cs}(\zeta)=\csc(\zeta)\pm\cot(\zeta),\ P=\frac{1}{4},\\
&Q=\frac{1-2m^2}{2},\ R=\frac{1}{4},\ m\to0,\\
&F(\zeta)=\mathrm{nc}(\zeta)\pm\mathrm{sc}(\zeta)=\sec(\zeta)\pm\tan(\zeta),\ P=\frac{1-m^2}{4},\\
&Q=\frac{1+m^2}{2},\ R=\frac{1-m^2}{4},\ m\to0.
\end{aligned} \tag{20}$$

From the balancing principle, (17) and (18) take the form:

$$P_1(\zeta)=A_0+A_1F(\zeta), \tag{21}$$

$$P_2(\zeta)=B_0+B_1F(\zeta)+B_2F^2(\zeta)\,. \tag{22}$$

Plugging (21) and (22) along with (19) into (11) and (12) results in

$$A_0 = 0,\ \ B_1 = 0,\ \ B_2 = \frac{\left(4a\kappa^2 - 4b\kappa\omega + 2\omega - c_2\right)QB_0}{\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)R},$$

$$\eta = \pm\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{2Q\left(-bv + a\right)}},\ A_1 = \pm\sqrt{-\frac{3P\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{Q\delta}},$$

$$d_1 = \frac{\delta\begin{pmatrix} 8PRa^2\kappa^4 - 16PRab\kappa^3\omega - 8PRa\kappa^2B_0c_1 + 4Q^2a\kappa^2B_0c_1 \\ + 8PRa\kappa^2\omega + 8PRb^2\kappa^2\omega^2 + 8PRb\kappa\omega B_0c_1 - 4Q^2b\kappa\omega B_0c_1 \\ - 8PRb\kappa\omega^2 + 2PRB_0^{\,2}c_1^{\,2} - 4PR\omega B_0c_1 + 2Q^2\omega B_0c_1 - Q^2B_0c_1c_2 + 2PR\omega^2 \end{pmatrix}}{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)^2 PR},$$

$$d_2 = -\frac{\delta B_0\begin{pmatrix} Q^2c_2^{\,2} - 2Q^2\omega c_2 + 3PR\omega^2 - 2Q^2B_0c_1c_2 + 4Q^2\omega B_0c_1 - 6PR\omega B_0c_1 \\ + 4Q^2b\kappa\omega c_2 - 4Q^2a\kappa^2c_2 + 3PRB_0^{\,2}c_1^{\,2} - 12PRb\kappa\omega^2 + 12PRa\kappa^2\omega \\ - 8Q^2b\kappa\omega B_0c_1 + 8Q^2a\kappa^2B_0c_1 + 12PRb\kappa\omega B_0c_1 - 12PRa\kappa^2B_0c_1 \\ + 12PRb^2\kappa^2\omega^2 - 24PRab\kappa^3\omega + 12PRa^2\kappa^4 \end{pmatrix}}{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)^2 PR}. \tag{23}$$

Inserting (23) into (21) and (22) yields dark solitons

$$q(x,t) = \pm\sqrt{\frac{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{2\delta}}\tanh\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4\left(bv - a\right)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \tag{24}$$

$$r(x,t) = \left\{B_0 - \frac{2\left(4a\kappa^2 - 4b\kappa\omega + 2\omega - c_2\right)B_0}{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}\tanh^2\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4\left(bv - a\right)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]\right\}e^{2i\left(-\kappa x + \omega t + \theta_0\right)}, \tag{25}$$

singular solitons

$$q(x,t) = \pm\sqrt{\frac{3\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{2\delta}}\coth\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4\left(bv - a\right)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \tag{26}$$

$$r(x,t) = \left\{B_0 - \frac{2\left(4a\kappa^2 - 4b\kappa\omega + 2\omega - c_2\right)B_0}{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}\coth^2\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{4\left(bv - a\right)}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right]\right\}e^{2i\left(-\kappa x + \omega t + \theta_0\right)}, \tag{27}$$

and combo singular solitons

$$q(x,t) = \pm\sqrt{\frac{6\left(2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega\right)}{4\delta}}\begin{pmatrix} \coth\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{bv - a}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right] \\ \pm\operatorname{csch}\left[\sqrt{\frac{2a\kappa^2 - 2b\kappa\omega - B_0c_1 + \omega}{bv - a}}\left(x - \frac{2b\omega - 4a\kappa}{1 - 2bk}t\right)\right] \end{pmatrix}^2 e^{i\left(-\kappa x + \omega t + \theta_0\right)}, \tag{28}$$

$$r(x,t)=\left\{\begin{array}{c} B_0-\dfrac{2\left(4a\kappa^2-4b\kappa\omega+2\omega-c_2\right)B_0}{2a\kappa^2-2b\kappa\omega-B_0c_1+\omega} \\ \times\left(\begin{array}{c}\coth\left[\sqrt{\dfrac{2a\kappa^2-2b\kappa\omega-B_0c_1+\omega}{bv-a}}\left(x-\dfrac{2b\omega-4a\kappa}{1-2bk}t\right)\right] \\ \pm\operatorname{csch}\left[\sqrt{\dfrac{2a\kappa^2-2b\kappa\omega-B_0c_1+\omega}{bv-a}}\left(x-\dfrac{2b\omega-4a\kappa}{1-2bk}t\right)\right]\end{array}\right)^2\end{array}\right\}e^{2i(-\kappa x+\omega t+\theta_0)}. \tag{29}$$

These soliton solutions are valid for

$$\delta\left(2a\kappa^2-2b\kappa\omega-B_0c_1+\omega\right)>0, \tag{30}$$

$$(bv-a)\left(2a\kappa^2-2b\kappa\omega-B_0c_1+\omega\right)>0. \tag{31}$$

2.2. Sine–Gordon equation approach

The solution structures of (11) and (12) according to this approach are taken to be

$$P_1(\zeta)=\sum_{i=1}^{N}\cos^{i-1}(V(\zeta))\left[B_i\sin(V(\zeta))+A_i\cos(V(\zeta))\right]+A_0, \tag{32}$$

$$P_2(\zeta)=\sum_{i=1}^{N}\cos^{i-1}(V(\zeta))\left[D_i\sin(V(\zeta))+C_i\cos(V(\zeta))\right]+C_0, \tag{33}$$

where A_i, B_i, C_i, and D_i for $1\le i\le N$ are constants, the number N is determined from balancing principle, and $V(\zeta)$ holds

$$V'(\zeta)=\sin(V(\zeta)). \tag{34}$$

Also, it needs to be mentioned that (34) has the following solutions:

$$\sin(V(\zeta))=\operatorname{sech}(\zeta)\ \text{ or }\ \sin(V(\zeta))=\operatorname{icsh}(\zeta),\ \cos(V(\zeta))=\tanh(\zeta)\ \text{ or }\ \cos(V(\zeta))=\coth(\zeta). \tag{35}$$

The balancing principle implies that

$$P_1(\zeta)=B_1\sin(V(\zeta))+A_1\cos(V(\zeta))+A_0, \tag{36}$$

$$P_2(\zeta)=\cos(V(\zeta))\left[D_2\sin(V(\zeta))+C_2\cos(V(\zeta))\right]+D_1\sin(V(\zeta))+C_1\cos(V(\zeta))+C_0. \tag{37}$$

Inserting (36) and (37) along with (34) into (11) and (12) leads to

$$\eta=\pm\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2(\delta-6d_1)(-bv+a)}},\ A_0=0,\ A_1=0,$$

$$B_1=\pm\sqrt{-\frac{9\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{\delta(\delta-6d_1)}},\ D_1=0,\ C_1=0,\ D_2=0,$$

$$C_0=-\frac{4a\delta^2\kappa^2-6a\delta\kappa^2d_1-4b\delta^2\kappa\omega+6b\delta\kappa\omega d_1+2\delta^2\omega-3\delta\omega d_1+3\delta c_1d_2-9c_1d_1d_2}{\delta(\delta-6d_1)c_1},$$

$$C_2=\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)(2\delta-3d_1)}{\delta(\delta-6d_1)c_1},\ c_2=4a\kappa^2-4b\kappa\omega+2\omega. \tag{38}$$

Substituting (38) along with (35) in (36) and (37) yields soliton solutions

$$q(x,t)=\pm\sqrt{-\frac{9\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{\delta\left(\delta-6d_1\right)}}\operatorname{sech}\left[\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\frac{2b\omega-4a\kappa}{1-2bk}t\right)\right]e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{39}$$

$$r(x,t)=\left\{\begin{array}{l}-\dfrac{4a\delta^2\kappa^2-6a\delta\kappa^2d_1-4b\delta^2\kappa\omega+6b\delta\kappa\omega d_1+2\delta^2\omega-3\delta\omega d_1+3\delta c_1d_2-9c_1d_1d_2}{\delta\left(\delta-6d_1\right)c_1}\\ +\dfrac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)\left(2\delta-3d_1\right)}{\delta\left(\delta-6d_1\right)c_1}\\ \times\tanh^2\left[\sqrt{-\dfrac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\dfrac{2b\omega-4a\kappa}{1-2bk}t\right)\right]\end{array}\right\}e^{2i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{40}$$

$$q(x,t)=\pm\sqrt{\frac{9\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{\delta\left(\delta-6d_1\right)}}\operatorname{csch}\left[\sqrt{-\frac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\frac{2b\omega-4a\kappa}{1-2bk}t\right)\right]e^{i\left(-\kappa x+\omega t+\theta_0\right)}, \tag{41}$$

$$r(x,t)=\left\{\begin{array}{l}-\dfrac{4a\delta^2\kappa^2-6a\delta\kappa^2d_1-4b\delta^2\kappa\omega+6b\delta\kappa\omega d_1+2\delta^2\omega-3\delta\omega d_1+3\delta c_1d_2-9c_1d_1d_2}{\delta\left(\delta-6d_1\right)c_1}+\\ +\dfrac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)\left(2\delta-3d_1\right)}{\delta\left(\delta-6d_1\right)c_1}\times\\ \times\coth^2\left[\sqrt{-\dfrac{3\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)}{2\left(\delta-6d_1\right)\left(-bv+a\right)}}\left(x-\dfrac{2b\omega-4a\kappa}{1-2bk}t\right)\right]\end{array}\right\}e^{2i\left(-\kappa x+\omega t+\theta_0\right)}. \tag{42}$$

Dark soliton (40) and singular soliton (42) are valid for

$$\left(\delta-6d_1\right)\left(-bv+a\right)\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)<0, \tag{43}$$

while bright soliton (39) is valid for the constraint (43) along with

$$\delta\left(\delta-6d_1\right)\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)<0 \tag{44}$$

and singular soliton (41) is valid for the constraint (43) along with

$$\delta\left(\delta-6d_1\right)\left(2a\delta\kappa^2-2b\delta\kappa\omega+\delta\omega+c_1d_2\right)>0. \tag{45}$$

3. Conclusions

Retrieved in this paper have been bright, dark, singular and combo singular embedded optical soliton solutions with the quadratic nonlinearity. A couple of integration schemes have been implemented to make this retrieval possible. The soliton solutions appeared with their respective existence criteria. Thus, the obtained results have paved its way to further future developments. One can handle embedded solitons using the variational principle as a sequel to previously reported results [5]. This time it needs to be studied using additional pulse formats. Another avenue to explore in this area is to handle the problem using the Lie symmetry analysis that will be a continuation and extension to previously recovered results [3]. These studies are under way, and their results will be soon published.

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Вбудовані солітони з $\chi^{(2)}$ і $\chi^{(3)}$ нелінійною сприйнятливістю

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Анотація. У цій роботі досліджено вбудовані солітони з квадратичною нелінійністю з урахуванням ефекту просторово-часової дисперсії. Обидві схеми інтегрування приводять до отримання яскравих, темних, сингулярних та комбінованих сингулярних солітонних розв’язків у неперервному режимі. Враховано також критерії існування цих солітонів.

Ключові слова: $\chi^{(2)}$ і $\chi^{(3)}$ нелінійності, вбудовані солітони.