Cubic-quartic optical soliton perturbation with Lakshmanan-Porsezian-Daniel model by semi-inverse variational principle

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Received: 20.03.2021

Abstract. We retrieve analytically a bright 1-soliton solution to a perturbed cubic–quartic Lakshmanan–Porsezian–Daniel model, using a semi-inverse variational principle. The perturbation terms are considered arising from the condition of maximum allowable intensity. The restrictions imposed by integrability considerations on the model parameters are enlisted. It is important that the other analytical approaches available fail to recover the analytical bright-soliton solution to the model with the maximum allowable intensity.

Keywords: cubic–quartic solitons, semi-inverse variation, stationary integrals

UDC: 535.32

1. Introduction

Lakshmanan–Porsezian–Daniel (LPD) equation is one of the most viable models that addresses dynamics of soliton propagation through waveguides [1, 2, 5]. It has been extensively studied and a number of important results have been recovered. These include stationary solitons [1], soliton solutions obtained using a sine-Gordon approach [5] and some others. Now it is time to take a look at cubic–quartic (CQ) solitons appearing in the presence of different Hamiltonian perturbation terms associated with a maximum allowable intensity. It is interesting that application of a semi-inverse variational principle can lead to 1-soliton solutions which cannot be obtained using any other integration schemes. The details of the semi-inverse variational principle, along with its successful application to the current model, are elucidated in the next sections, after a brief introduction to the governing model.

A dimensionless structure of the LPD model with CQ solitons (abbreviated as CQ–LPD further on) in the presence of perturbation terms is given by [5]

$$\begin{aligned} & iq_t + iaq_x xx + bq_x xxx + c\,|\,q\,|^2 \\ & = \alpha\left(q_x\right)^2 q^* + \beta\left|q_x\right|^2 q + \gamma\left|q\right|^2 q_{xx} + \lambda q^2 q_{xx}^* + \delta\left|q\right|^4 q + i\left[\theta_1\left(\left|q\right|^{2m} q\right)_x + \theta_2\left(\left|q\right|^{2m}\right)_x q + \theta_3\left|q\right|^{2m} q_x\right], \end{aligned} \quad (1)$$

where t and x are respectively temporal and spatial variables, $q(x,t)$ represents the wave profile, and $i = \sqrt{-1}$. The first term on the l. h. s. describes the linear temporal evolution, while a and b are the coefficients linked with the third-order and fourth-order dispersion terms, respectively. The coefficient c describes the Kerr law for the nonlinear refractive index. On the r. h. s. of Eq. (1), the coefficients α, β, γ and λ originate from the perturbation terms that come from the nonlinear dispersion, and δ corresponds to the two-photon absorption. The Hamiltonian perturbation terms are associated with the coefficients θ_j (j = 1, 2 and 3). Here θ_1 governs the self-steepening effect, and θ_2 and θ_3 are due to the nonlinear dispersion effects. Finally, the parameter m corresponds to the maximum allowable light intensity.

The linear dispersion terms that stem from the third- and fourth-order dispersions constitute the CQ solitons. These dispersion effects are obtained after replacing the chromatic dispersion, due to its low count, by the combined third- and fourth-order dispersions. Hence, the CQ solitons associated with the LPD model start touching upon the solitons that arise from the chromatic dispersion in the same model. Such a novel concept has recently gained considerable attention, as has been reflected in the recent reports (see Ref. [5]). It is not out of place to point out that, in the past, the LPD model with the chromatic dispersion has already been addressed using the semi-inverse variational principle [2]. Therefore, in the current work we study the same model, though with the chromatic dispersion replaced collectively by the third- and fourth-order dispersions.

2. Mathematical analysis

To start with, let us set

$$q(x,t) = g(s)e^{i\psi(x,t)}, \tag{2}$$

which is the phase–amplitude decomposition of the complex-valued function $q(x,t)$. Here

$$s = x - vt \tag{3}$$

forms the amplitude portion, with v indicating the soliton velocity. The phase component is given by

$$\phi(x,t) = -\kappa x + \omega t + \theta_0 . \tag{4}$$

The parameters κ, ω, θ_0 and v are the wave number, the frequency and the phase constant of soliton. Substituting Eq. (2) into Eq. (1) and decomposing it into real and imaginary parts yields a pair of relations. In particular, the imaginary part yields in the following:

$$\begin{aligned}&(a-4b\kappa)g'' - (v+3a\kappa^2-4b\kappa^3)g' \\ &+\{(2m+1)\theta_1+2m\theta_2+\theta_3+2\kappa(\alpha+\gamma-\lambda)\}g^2g' = 0\end{aligned}. \tag{5}$$

Here g' stands for dg/ds, g'' for d^2g/ds^2, and so on. Setting the coefficients entering in the linearly independent functions to be zero results in the soliton speed,

$$v = -3a\kappa^2 + 4b\kappa^3 . \tag{6}$$

Moreover, a pair of parameter constraints emerge:

$$a = 4b\kappa \tag{7}$$

and

$$2\kappa(\alpha+\gamma-\lambda)+(2m+1)\theta_1+2m\theta_2+\theta_3 = 0 . \tag{8}$$

Next, the real part of Eq. (1) gives

$$\begin{aligned}&bg^{(iv)} + (3a\kappa-6b\kappa^2)g'' - (\gamma+\lambda)g^2g'' - (\alpha+\beta)gg' - (\omega+a\kappa^3-b\kappa^4)g \\ &+\{c+(\alpha-\beta+\gamma+\lambda)\kappa^2\}g^3 - \delta g^5 + \kappa(\theta_1+\theta_3)g^{2m+1} = 0.\end{aligned} \tag{9}$$

For integrability, we put

$$\alpha+\beta=0 \tag{10}$$

and

$$\gamma+\lambda=0\,. \tag{11}$$

This reduces Eq. (9) to the form

$$\begin{aligned} &bg^{(iv)}+\left(3a\kappa-6b\kappa^2\right)g''-\left(\omega+a\kappa^3-b\kappa^4\right)g \\ &+\left\{c+(\alpha-\beta+\gamma+\lambda)\kappa^2\right\}g^3-\delta g^5+\kappa\left(\theta_1+\theta_3\right)g^{2m+1}=0 \end{aligned}. \tag{12}$$

Multiplication of Eq. (12) by g', integration with respect to the variable s and subsequent simplification reduces it to

$$\begin{aligned} &6b\left(g''\right)^2-18\kappa\left(a-2b\kappa\right)\left(g'\right)^2+6\left(\omega+a\kappa^3-b\kappa^4\right)g^2 \\ &-3\left\{c+(\alpha-\beta)\kappa^2\right\}g^4+2\delta g^6+\frac{6\kappa\left(\theta_1+\theta_3\right)}{m+1}g^{2m+2}=K \end{aligned}, \tag{13}$$

where K denotes an integration constant. The stationary integral is now defined as

$$\begin{aligned} J&=\int_{-\infty}^{\infty}K ds \\ &=\int_{-\infty}^{\infty}\left[\begin{aligned} &6b\left(g''\right)^2-18\kappa\left(a-2b\kappa\right)\left(g'\right)^2+6\left(\omega+a\kappa^3-b\kappa^4\right)g^2 \\ &-3\left\{c+(\alpha-\beta)\kappa^2\right\}g^4+2\delta g^6+\frac{6\kappa\left(\theta_1+\theta_3\right)}{m+1}g^{2m+2} \end{aligned}\right]ds \end{aligned}. \tag{14}$$

3. Application of semi-inverse variational principle

The semi-inverse variational principle states that the bright-soliton solution of the perturbed CQ–LPD Eq. (1) would bear the same form as its unperturbed counterpart upon setting $\theta_j=0$ at $j=1$, 2 and 3. However, the soliton amplitude A and its inverse width B for the perturbed version are determined from a governing pair of relations,

$$\frac{\partial J}{\partial A}=0 \tag{15}$$

and

$$\frac{\partial J}{\partial B}=0\quad . \tag{16}$$

Next, the amplitude part of the homogeneous CQ–LPD equation is structured as [5]

$$g(s)=A\ sech(Bs)\,. \tag{17}$$

Substitution of Eq. (17) into Eq. (14) and integration lead to

$$\begin{aligned} J=&\frac{28b}{5}A^2B^3-12\kappa\left(a-2b\kappa\right)A^2B+12\left(\omega+a\kappa^3-b\kappa^4\right)\frac{A^2}{B} \\ &-4\left\{c+(\alpha-\beta)\kappa^2\right\}\frac{A^4}{B}+\frac{32\delta}{15}\frac{A^6}{B}+\frac{12\kappa\left(\theta_1+\theta_3\right)G}{2m+1}\frac{A^{2m+1}}{B} \end{aligned}, \tag{18}$$

where

$$G=\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}\,. \tag{19}$$

From the stationary integral, the coupled system of Eqs. (15) and (16) can be reformulated as

$$\begin{aligned}&7bB^4-15\kappa(a-2b\kappa)B^2+15(\omega+a\kappa^3-b\kappa^4)\\&-10\{c+(\alpha-\beta)\kappa^2\}A^2+8\delta A^4+\frac{15(m+1)\kappa(\theta_1+\theta_3)G}{2m+1}A^{2m}=0\end{aligned} \tag{20}$$

and

$$\begin{aligned}&63bB^4-45\kappa(a-2b\kappa)B^2-45(\omega+a\kappa^3-b\kappa^4)\\&+15\{c+(\alpha-\beta)\kappa^2\}A^2-8\delta A^4-\frac{45(m+1)\kappa(\theta_1+\theta_3)G}{2m+1}A^{2m}=0\end{aligned}. \tag{21}$$

Upon uncoupling, a relation biquadratic in B emerges:

$$84bB^4-90\kappa(a-2b\kappa)B^2-15\{c+(\alpha-\beta)\kappa^2\}A^2+16\delta A^4+HA^{2m}=0\ , \tag{22}$$

where

$$H=\frac{3m(m+1)\kappa(\theta_1+\theta_3)G}{2m+1}\ . \tag{23}$$

Eq. (23) reveals the relationship between the soliton amplitude and its width as

$$B=\left[\frac{45\kappa(a-2b\kappa)+\sqrt{2025\kappa^2(a-2b\kappa)^2+84b\left[15\{c+(\alpha-\beta)\kappa^2\}A^2-16\delta A^4-HA^{2m}\right]}}{84b}\right]^{\frac{1}{2}}. \tag{24}$$

The mathematical structure of Eq. (24) poses a couple of parameter constraints for the bright soliton to exist. They are as follows:

$$2025\kappa^2(a-2b\kappa)^2+84b\left[15\{c+(\alpha-\beta)\kappa^2\}A^2-16\delta A^4-HA^{2m}\right]>0 \tag{25}$$

and

$$b\left[45\kappa(a-2b\kappa)+\sqrt{2025\kappa^2(a-2b\kappa)^2+84b\left[15\{c+(\alpha-\beta)\kappa^2\}A^2-16\delta A^4-HA^{2m}\right]}\right]>0\,. \tag{26}$$

Hence, the bright 1-soliton solution to the CQ–LPD equation with the maximum allowable intensity is given by

$$q(x,t)=A\ sech\left[B(x-vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{27}$$

where the amplitude−width relation for the soliton is given by Eq. (25) and the soliton velocity by Eq. (6). The parameter constraints imposed in order that the solitons existed are given by Eqs. (7), (8), (26) and (27).

4. Conclusion

The present work has recovered the bright 1-soliton solution of the perturbed CQ–LPD equation, in which the perturbation terms are those laid by the maximum allowable intensity. It is important that this analytical soliton solution cannot be retrieved by any other available analytical schemes at the arbitrary exponents of intensity. Our solution is not exact since it is based on the semi-inverse variational principle. Nevertheless, this solution would be interesting for the telecommunication industry whenever the chromatic dispersion is minimal and so can be discarded. We have also enlisted the constraints imposed on the model parameters, which emerge naturally from the very structure of the solution. Our results are going to be of great value for the Internet communication industry.

Acknowledgement

The research work of the sixth author (M.R.B) was supported by the grant NPRP 11S-1246-170033 from QNRF. The authors also declare that there is no conflict of interest.

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Biswas A, Edoki J, Guggilla P, Khan S, Alzahrani A K and Belic M R. 2021. Cubic–quartic optical solitons in Lakshmanan–Porsezian–Daniel model derived with semi-inverse variational principle. Ukr.J.Phys.Opt. **22**: 123 – 127. doi: 10.3116/16091833/22/3/123/2021

Анотація. *Використовуючи напівобернений варіаційний принцип, аналітично одержано яскравий 1-солітонний розв'язок для збуреної кубічно-квартичної моделі Лакшманана–Порсезіана–Даніеля. Розглянуто члени збурення, що виникають із вимоги гранично допустимої інтенсивності. Описано обмеження, накладені міркуваннями інтегрованості на параметри моделі. Важливо, що всі інші наявні аналітичні підходи не змогли відтворити аналітичний розв'язок для яскравого солітона для даної моделі з гранично допустимою інтенсивністю.*