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Propagation of chirped gray optical dips in nonlinear metamaterials

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A B S T R A C T

We investigate the existence and stability properties of nonlinearly chirped solitary waves on a continuous-wave background in nonlinear metamaterials with higher-order effects such as pseudo-quintic nonlinearity and self-steepening effect. Novel classes of chirped gray solitary pulses (dark pulses with nonzero minimum intensity) are derived by employing the traveling-wave method. The conditions for the formation of these localized pulses as well as the nonlinear chirp associated to them are presented. The stability of the obtained structures is also demonstrated numerically under finite initial perturbations. The results show that the envelope pulses obtained here represent new types of extremely robust nonlinear chirped structures in highly nonlinear metamaterials.

1. Introduction

The *optical soliton* represents a nonlinear pulse or wave packet which travels without changing shape over extremely large distances. Such a pulse can occur in single-mode optical fibers when the pulse broadening of the group-velocity dispersion (GVD) and the compressing of the Kerr nonlinearity are in balance [1]. After the first theoretical prediction [2,3] and the experimental demonstration [4] of the temporal soliton in a monomode optical fiber, interest has been aroused in the studies of such pulses over the past decades since it may be applied as bit rates in the next generation of optical communication systems [5–8].

Normally in nonlinear optical fibers, the propagation of soliton pulses is governed by the nonlinear Schrödinger (NLS) equation which is integrable by the inverse scattering method [9]. Depending on the relative signs of linear group-velocity dispersion and nonlinearity induced self-phase modulation, such a model admits two distinct types of localized solutions, called bright and dark soliton solutions [10,11]. However, when the spectral width of the pulses becomes comparable with the carrier frequency, additional higher-order effects such as the self-steepening otherwise called the Kerr dispersion, the Raman stimulated scattering (soliton self-frequency shift), and third-order linear dispersion should be taken into account [12]. Due to these additional effects, the governing higher-order NLS equation can lead to a soliton behavior applicable to a subpicosecond or femtosecond regime. Propagations of such envelope solitons in optical fibers are of particular interest because of their advantages in long distance, high bit rate, and faster switching [13,14].

Recent advances in the field of solitons have shown that many new localized structures can be formed in nonlinear materials. For instance, it has been demonstrated that the exact balance between dispersion/diffraction and nonlinearity can induce solitons characterized by fascinating shape-preserving wave phenomena such as Gaussian soliton clusters, multipole soliton clusters, nested soliton clusters [15], and light bullets (spatiotemporal solitons) [16]. Peregrine and breather structures have been also reported in a medium with partially nonlocal inhomogeneous nonlinearities [17]. Additionally, Gaussian spatial solitons have been interestingly obtained in quintic–septimal nonlinear materials under PT-symmetric potentials [18].

In recent years, the femtosecond gray solitary waves or solitons (which are dark pulses with nonzero minimum intensity) have been the objects of extensive theoretical and experimental studies [19–23].

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Unlike the bright soliton, where the center point is a maximum, the center point for the dark and gray solitons is located at the minimum in amplitude [21]. Until now, gray pulses propagation in nonlinear metamaterials has been analyzed only for the case of chirp-free and linearly chirped solitary waves or solitons (see, e.g., Refs. [22,23]). Particularly interesting are the so-called *chirped pulses* which are useful in the design of optical devices such as fiber-optic amplifiers, optical pulse compressors and solitary wave based communication links [24,25]. To our knowledge, the dynamics and stability of nonlinearly chirped gray solitons in nonlinear metamaterials have not been reported yet.

In this paper, we investigate for the first time to our knowledge the propagation and stability properties of nonlinearly chirped gray solitary pulses in nonlinear metamaterials exhibiting higher-order effects such as pseudo-quintic nonlinearity and self-steepening effect. Novel classes of chirped gray solitary pulses are derived using the traveling-wave method. It is remarked that the self-steepening effect could make the envelope pulses chirped, and the nonlinear chirp associated with these pulses has a nontrivial form which includes two intensity dependent chirping terms apart from the linear contribution.

The article is organized as follows. In Section 2, the theoretical model describing the propagation of ultrashort pulses in metamaterials with non-Kerr nonlinearity is presented. Particular cases of the governing equation which describe the wave dynamics in a variety of physical systems are also discussed. In Section 3, we present two new types of nonlinearly chirped gray solitary waves as well as the nonlinear chirp associated with these optical pulses. Section 4 is devoted to the stability analysis of such privileged exact solutions under some perturbations. In Section 5, we present the three basic conserved quantities for the model. Our findings are summarized in Section 6.

2. The model

The generalized NLS equation describing the evolution of femtosecond optical field ψ in nonlinear metamaterials can be written as [23,26]

$$i\frac{\partial\psi}{\partial z}+\frac{k_2}{2}\frac{\partial^2\psi}{\partial t^2}+p_3|\psi|^2\psi-p_5|\psi|^4\psi-is_1\frac{\partial}{\partial t}\left(|\psi|^2\psi\right)=0, \tag{1}$$

where $t = cT/\lambda_p$ and $z = Z/\lambda_p$ are the respective normalized time and propagation distance, with λ_p is the plasma wavelength. Also k_2 stands for the GVD coefficient, s_1 represents the self-steepening coefficient, while p_3 and p_5 represent cubic and pseudo-quintic nonlinear coefficients respectively.

This equation contains several particular cases such as the standard NLS equation ($p_5 = s_1 = 0$) [27], the modified NLS equation ($p_5 = 0$) [28], the Kaup–Newell equation ($p_3 = p_5 = 0$) [29], the cubic–quintic NLS equation ($s_1 = 0$) [30], and the pure quintic NLS equation ($p_3 = s_1 = 0$) [31]. Each particular case is important to describe nonlinear wave dynamics in specific physical systems. In a recent work, exact quasi-soliton solutions for this equation have been derived under the condition of vanishing self-steepening term ($s_1 = 0$) [23]. Moreover, the modulational instability and localization of an ultrashort electromagnetic pulse that is governed by the generalized NLS equation above have been examined by Marklund et al. [26]. Here we concentrate on discussing it with nonvanishing boundary conditions by using the traveling-wave method.

3. Exact chirped gray soliton solutions

The richness of an optical medium that is actively influenced by ultrashort pulses propagation is often measured by the variety of structures that it can support. Of particular interest are exact and new-type solutions that describe the evolution of nonlinearly chirped solitons on a cw background. In order to obtain such structures we proceed as in [32–34] and express the complex envelope function $\psi(z,t)$ as

$$\psi(z,t)=\rho(\xi)\,e^{i[\chi(\xi)-\kappa z]}, \tag{2}$$

where $\xi = t - uz$ is the traveling coordinate, $\rho(\xi)$ and $\chi(\xi)$ are real functions of ξ, and κ is the wave number constant ($\kappa > 0$). Also, $u = 1/\mathrm{v}$, with v is the group velocity of the wave packet. The corresponding chirping is given by $\delta\omega(z,t) = -\frac{\partial}{\partial t}\left[\chi(\xi)-\kappa z\right] = -\chi'(\xi)$. Substituting the representation (2) into Eq. (1) and separating real and imaginary parts of the resulting equation, one derives the coupled equations in ρ and χ,

$$\kappa\rho+u\chi'\rho+\frac{k_2}{2}\left(\rho''-\chi'^2\rho\right)+p_3\rho^3-p_5\rho^5+s_1\chi'\rho^3=0, \tag{3}$$

and

$$-u\rho'+\frac{k_2}{2}\left(\rho\chi''+2\chi'\rho'\right)-3s_1\rho^2\rho'=0, \tag{4}$$

where the prime indicates differentiation with respect to ξ.

Multiplying both sides of (4) by ρ and integrating leads to:

$$\chi'=\frac{3s_1}{2k_2}\rho^2+\frac{A}{k_2\rho^2}+\frac{u}{k_2}, \tag{5}$$

where A is an integration constant to be determined latter. The corresponding chirping takes the form:

$$\delta\omega(z,t)=-\frac{3s_1}{2k_2}\rho^2-\frac{A}{k_2\rho^2}-\frac{u}{k_2}. \tag{6}$$

Thus, we see from the result above that the chirping has a nontrivial structure which contains two intensity dependent contributions apart from the linear term (evidently $I = \rho^2$ being the intensity). We further observe that the first nonlinear contribution is directly proportional to the intensity of the wave and depends on the self-steepening parameter while the second one is inversely proportional to the intensity.

Further substitution of the expression (5) into (3) gives the following evolution equation:

$$\rho''+\frac{2\kappa k_2+u^2-s_1A}{k_2^2}\rho+\frac{2\left(p_3k_2+s_1u\right)}{k_2^2}\rho^3+\frac{3s_1^2-8p_5k_2}{4k_2^2}\rho^5-\frac{A^2}{k_2^2\rho^3}=0, \tag{7}$$

which describes the dynamics of the wave amplitude in the nonlinear metamaterial. Multiplying (7) by ρ' and integrating once with respect to ξ, one obtains

$$\left(\rho'\right)^2+\frac{2\kappa k_2+u^2-s_1A}{k_2^2}\rho^2+\frac{p_3k_2+s_1u}{k_2^2}\rho^4+\frac{3s_1^2-8p_5k_2}{12k_2^2}\rho^6+\frac{A^2}{k_2^2\rho^2}+2B=0, \tag{8}$$

where B is the second integration constant.

More physical insight into the competition between GVD, cubic–quintic nonlinearities, and self-steepening can be obtained by an effective particle analogy. Applying the transformation $F = \rho^2$ to Eq. (8), we easily obtain the potential-well description:

$$\left(\frac{dF}{d\xi}\right)^2+V(F)=0, \tag{9a}$$

where the potential V is

$$V(F)=\delta+\alpha F+\beta F^2+\gamma F^3+\nu F^4, \tag{9b}$$

with

$$\delta=\frac{4A^2}{k_2^2},\ \alpha=8B,\ \beta=\frac{4\left(2\kappa k_2+u^2-s_1A\right)}{k_2^2},\quad \gamma=\frac{4\left(p_3k_2+s_1u\right)}{k_2^2},\ \nu=\frac{3s_1^2-8p_5k_2}{3k_2^2}. \tag{9c}$$

Eq. (9a) resembles the equation describing the dynamics of a particle in a potential well $V(F)$. Below, we present novel chirped solitary wave solutions on a cw background of Eq. (1) based on solving this equation.

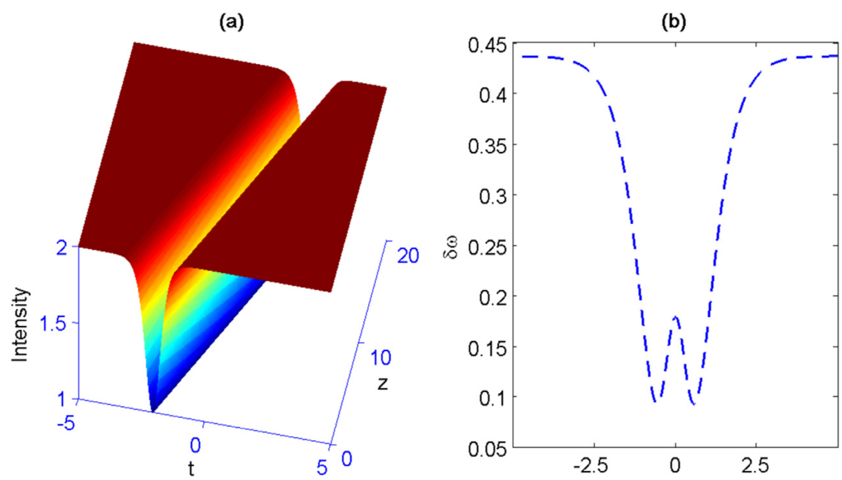

Fig. 1. (a) Evolution of intensity wave profile of the gray solitary wave as computed from (14) of Eq. (1) and (b) profile of chirping given by Eq. (15). Here we have used the parameters values $k_2 = -0.7954$, $p_3 = -1.2566\times10^{-10}$, $p_5 = -2.095\times10^{-21}$, $s_1 = 0.6666\times10^{-10}$, $\lambda = 2\times10^{10}$, $\sigma = 10^{10}$, $\eta = 1$ and $u = -2.4484$. The solitary wave intensity is normalized by $|\psi(z,t)|^2/\sigma$.

The nonlinear chirp associated with each of these optical pulses as well as the conditions for their existence are also determined. Here we present the exact chirped gray solitary wave solutions of Eq. (1) in two cases.

Case-1. The first type of exact solitary wave solution we obtained here for Eq. (9) takes the form

$$F(\xi) = \lambda - \sigma\,\mathrm{sech}^2(\eta\xi), \tag{10}$$

for the special physical condition:

$$s_1^2 = \frac{8k_2p_5}{3}, \tag{11}$$

between the self-steepening parameter, GVD and pseudo-quintic nonlinearity. For this case, the relationship between different parameters of the nonlinear equation for the squared wave amplitude (9a) and the parameters of the solution satisfy

$$\delta = \frac{4\lambda^2\eta^2(\lambda-\sigma)}{\sigma},\quad \alpha = \frac{4\lambda\eta^2(2\sigma-3\lambda)}{\sigma},$$
$$\beta = \frac{4\eta^2(3\lambda-\sigma)}{\sigma},\quad \gamma = -\frac{4\eta^2}{\sigma}. \tag{12}$$

Equating the parameters in (9) and (12), we get the expressions of the inverse group velocity u and wave number κ:

$$u = -\frac{\eta^2k_2^2 + p_3k_2\sigma}{\sigma s_1}, \tag{13a}$$
$$\kappa = \frac{\eta^2k_2^2(3\lambda-\sigma) + \left(s_1A - u^2\right)\sigma}{2k_2\sigma}, \tag{13b}$$

together with the expressions of A and B:

$$A^2 = \frac{k_2^2\lambda^2\eta^2(\lambda-\sigma)}{\sigma}, \tag{13c}$$
$$B = \frac{\lambda\eta^2(2\sigma-3\lambda)}{2\sigma}, \tag{13d}$$

From the relations (13c) and (13d), we notice that the integration constants A and B can be fixed easily with the parameters of the pulse, and consequently the physical meaning of them is very clear. One can also see that the constant A in (13c) takes a real value for $\lambda > \sigma$ if considering $\sigma > 0$.

Thus, the exact chirped solitary pulse solution on a cw background to the generalized NLS equation (1) is given by

$$\psi(z,t) = \sqrt{\lambda - \sigma\,\mathrm{sech}^2[\eta(t-uz)]}\,e^{i[\chi(\xi)-\kappa z]} \tag{14}$$

with $\lambda > \sigma$. Here the parameter λ decides the strength of the background, in which this solution propagates in the nonlinear metamaterial. It is worth observing that this family of solutions has three free parameters λ, σ and η if the metamaterial parameter k_2, p_3, and p_5 are known. Of course the self-steepening parameter s_1 will be determined using the existence condition (11). The latter indicates that the GVD and pseudo-quintic nonlinearity must satisfy the condition $k_2p_5 > 0$, which implies that the obtained solitary wave can exist in abnormal (normal) dispersion for self-defocusing (-focusing) nonlinearity [see Eq. (11)]. Naturally, the solution (14) reduces to a dark-type soliton in the limit $\lambda = \sigma$.

The corresponding chirping associated with this nonlinearly chirped solitary wave can be obtained readily as

$$\delta\omega(z,t) = -\frac{3s_1}{2k_2}\left(\lambda - \sigma\,\mathrm{sech}^2(\eta\xi)\right) - \frac{A}{k_2\left(\lambda - \sigma\,\mathrm{sech}^2(\eta\xi)\right)} - \frac{u}{k_2}. \tag{15}$$

Fig. 1(a) shows the evolution of the intensity wave profile of the preceding chirped solitary wave solution of Eq. (1) for the parameters values [23]: $k_2 = -0.7954$, $p_3 = -1.2566\times10^{-10}$, and $p_5 = -2.095\times10^{-21}$. To satisfy the parametric condition (11), we set $s_1 = 0.6666\times10^{-10}$. Here the other solitary wave parameters are taken as $\lambda = 2\times10^{10}$, $\sigma = 10^{10}$ and $\eta = 1$. As concerns the pulse inverse group velocity u, it can be obtained from Eq. (13a) as $u = -2.4484$. As one can see from Fig. 1(a), this solution represents a chirped gray pulse with a nonzero dip originating from the higher-order nonlinear effects. The chirping profile for this gray-type solution is shown in Fig. 1(b) (for $z = 0$). One can see that it takes the shape of W and saturates at the same finite value as $t \to \pm\infty$.

It is interesting to note that for the special case of $A = B = 0$, we have $\lambda = 0$ and our solution (14) simply reduces to the waveform:

$$\psi(z,t) = ae^{i\Gamma\tanh[\eta(t-uz)]}$$
$$\times\,\mathrm{sech}[\eta(t-uz)]\,e^{i[Kz-\Omega t+\varphi]} \tag{16}$$

which is exactly the solitary wave solution found in Ref. [35]. Here φ is a constant phase while $a = \sqrt{-\sigma}$, $\eta^2 = -\frac{2\kappa k_2+u^2}{k_2^2}$, $\Gamma = \frac{3s_1a^2}{2k_2\eta}$, $K = -\frac{u^2}{k_2} - \kappa$, $\Omega = -\frac{u}{k_2}$ with $\sigma = -\frac{\eta^2k_2^2}{p_3k_2+s_1u}$.

Case-2. It is of further interest to search for other types of exact chirped localized solutions because closed form solutions are helpful for recognizing physical phenomena described by the model. The second type of exact solitary wave solution we find here for Eq. (9) is of the form

$$F(\xi) = \frac{\lambda^2}{1 + r\,\mathrm{sech}^2(\mu\xi)}, \tag{17}$$

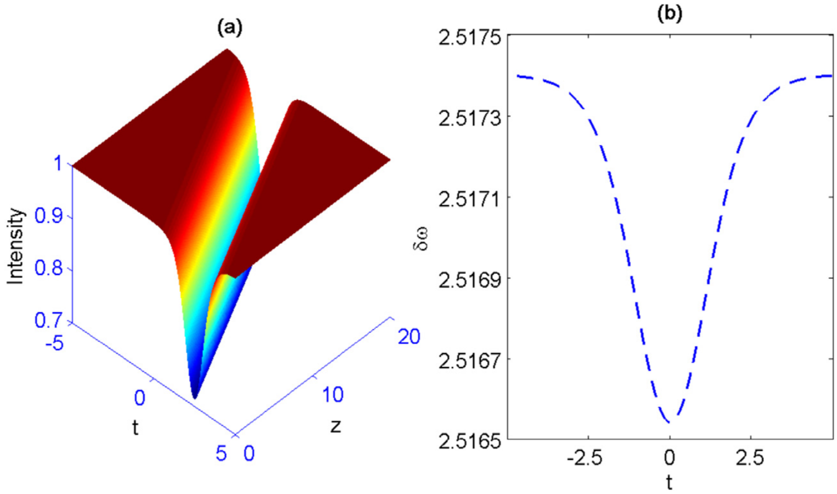

Fig. 2. (a) Evolution of intensity wave profile of the gray solitary wave as computed from (20) of Eq. (1) and (b) profile of chirping given by Eq. (21). Here we have used the parameters values $k_2 = -0.7954$, $p_3 = -1.2566 \times 10^{-10}$, $p_5 = -2.095 \times 10^{-21}$, $s_1 = 4.8195 \times 10^{-14}$, $r = 0.4$ and $u = 2$. The solitary wave intensity is normalized by $|\psi(z,t)|^2/|\lambda|^2$.

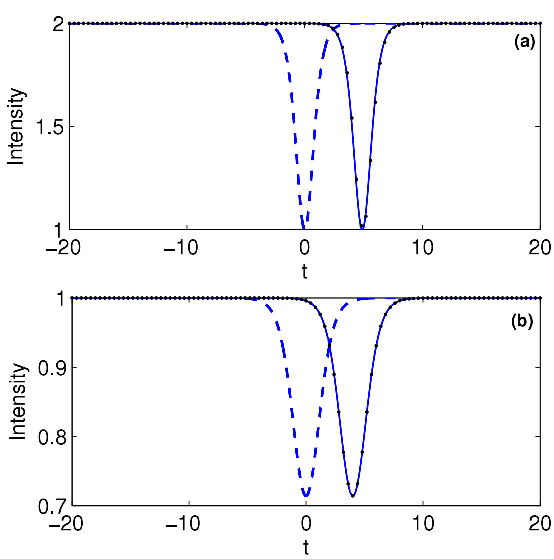

Fig. 3. Comparison of our analytical results (a) given by Eq. (14) and (b) Eq. (20) at $z = 20$ (solid line) with the numerical simulations (circle) and the initial profile at $z = 0$ (dotted line).

where

$$\mu^2 = -\frac{\gamma^2 r(r+1)}{4\nu(2r+3)^2}, \tag{18a}$$

$$\lambda^2 = -\frac{\gamma(r+1)}{\nu(2r+3)}, \tag{18b}$$

and

$$\alpha = \frac{\gamma^3(r+1)^2}{\nu^2(2r+3)^3}, \quad \beta = \frac{\gamma^2(r+1)(r+3)}{\nu(2r+3)^2}, \quad \delta = 0. \tag{18c}$$

Equating (9c) and (18c), one finds that the integration constant A vanishes while the constant B and the wave number κ are given, in turn, by

$$B = \frac{\gamma^3(r+1)^2}{8\nu^2(2r+3)^3}, \tag{19a}$$

and

$$\kappa = \frac{k_2^2\gamma^2(r+1)(r+3) - 4u^2\nu(2r+3)^2}{8k_2\nu(2r+3)^2}. \tag{19b}$$

With these results, we find that the exact chirped solitary wave solution of the generalized NLS equation (1) can be written as

$$\psi(z,t) = \frac{\lambda}{\sqrt{1 + r\,\mathrm{sech}^2(\mu\xi)}} e^{i[\chi(\xi) - \kappa z]}. \tag{20}$$

with the primary requirement $r > 0$. Because the pulse width parameter μ needs to be real for the existence of the chirped solution (20), one must require $\nu < 0$ in Eq. (18a), which implies $s_1^2 < 8p_5k_2/3$. An interesting observation is that the solution (20) is located on a nonzero background with an intensity value $|\psi|^2 = \lambda^2/(1+r)$ at the center of the pulse, and $|\psi|^2 = \lambda^2$ when $\xi \to \pm\infty$. It is worth noting here that this solution may represent a chirped bright-type solitary wave solution on a constant background for $-1 < r < 0$.

Then, the corresponding chirping takes the form

$$\delta\omega(z,t) = -\frac{3s_1\lambda^2}{2k_2\left(1 + r\,\mathrm{sech}^2(\mu\xi)\right)} - \frac{u}{k_2}. \tag{21}$$

The evolution of the intensity wave profile of the gray solitary wave solution (20) obtained in this case for Eq. (1) and chirping profile (for $z = 0$) are shown in Fig. 2(a) and 2(b), respectively. The parameters used here are [23]: $k_2 = -0.7954$, $p_3 = -1.2566\times10^{-10}$, and $p_5 = -2.095\times10^{-21}$. The self-steepening parameter is taken as $s_1 = 4.8195 \times 10^{-14}$. The other parameters are taken as $r = 0.4$ and $u = 2$. It is clear from Fig. 2(a) that this nonlinearly chirped solution describes a gray pulse with a nonzero dip. Chirping for this gray-type solution (depicted in Fig. 2(b)) has a minimum at the center of the pulse and saturates at the same finite value as $t \to \pm\infty$.

Comparing the analytical results (14) and (20) with numerical solutions in Figs. 1 and 2, we infer that these exact chirped gray solitary waves can exist in nonlinear metamaterials, and that the higher-order effects not only play a significant role for the formation of gray solitary pulses but also dominate their propagations in metamaterials.

In what follows, for the completeness of the investigation, we solve the underlying Eq. (1) numerically by applying the split-step Fourier method [36]. Here, the exact solitary wave solutions (14) and (20) at $z = 0$ are chosen as initial pulses. Fig. 3(a) and (b), respectively, show a comparison of the analytical results (14) and (20) with the numerical simulations at $z = 20$. The choice of metamaterial and pulse parameters are the same as those in Figs. 1(a) and 2(a), respectively. It is clear that the numerical results agree very well with the analytical solutions as illustrated in Fig. 3. Note that the profiles or shapes of pulses are well preserved after propagating the distance simulated in the metamaterial.

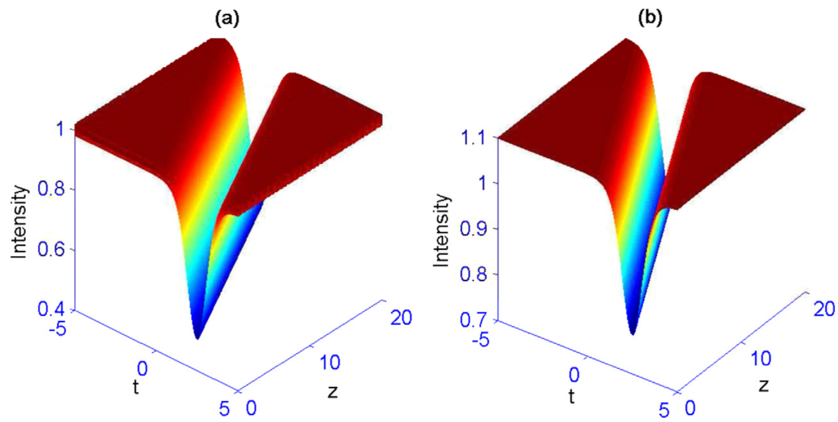

Fig. 4. Evolution of intensity wave profile of the gray solitary wave as computed from (14) (a) under the perturbation of white noise with the same parameters as in Fig. 1(a); (b) evolution of an initial pulse whose amplitude is 10% smaller than the theoretical prediction.

4. Stability analysis

An important problem is the stability of the new nonlinearly chirped pulses presented above against finite perturbations. It is worth noting that only stable (or weakly unstable) solitary waves can be observed experimentally [37,38]. In general, bright soliton solutions to the one-dimensional cubic–quintic NLS equation are known to be remarkably stable [39]. Moreover, bright, dark and gray nonautonomous soliton solutions for the cubic–quintic NLS equation with distributed coefficients are also found to be stable under some initial perturbation while evolving in distance [30]. Since the model under consideration (1) includes such cubic–quintic nonlinearities, as well as the self-steepening term, it is reasonable to conjecture that the chirped solitary wave solutions (14) and (20) should be stable. However, in order to confirm the stability of these localized structures, one have to study the numerical evolution of the initial optical pulses under some perturbations (e.g., the perturbation of the amplitude, random noises, and the slight violation of the parametric conditions).

In the following, we demonstrate the stability of solutions with respect to finite perturbations by performing various types of numerical experiments. Here we take the nonlinearly chirped pulse solution (14) as an example and perform two types of direct numerical simulations with initial white noise and amplitude perturbation to study the stability of this solution compared to Fig. 1(a). First, we add 10% white noise in the initial pulse [40]. Second, we perturb the amplitude (10%) in the initial distribution [30]. The numerical results are shown in Fig. 4(a) in which the solutions are affected by the random noise and Fig. 4(b) in which the initial amplitude is perturbed. The results reveal that finite initial perturbations (10%), such as amplitude and white noise, could not influence the main character of the solution.

5. Conservation laws

The last topic that this paper will touch base on is the conservation laws of the model. Without a discussion on conserved quantities, no study on optical solitons is complete. Therefore, to keep the paper rounded, it is imperative to discuss the conserved quantities for the model. The three immediate conservations laws for the governing model (1) are power (P), linear momentum (M) and Hamiltonian (H) and they are respectively given by:

$$P = \int_{\infty}^{\infty} |\psi|^2 dt, \tag{22}$$

$$M = ik_2 \int_{\infty}^{\infty} \left(\psi^* \psi_t - \psi \psi_t^*\right) dt, \tag{23}$$

and

$$H = \int_{\infty}^{\infty} \left(3k_2 |\psi_t|^2 - 3p_3 |\psi|^4 + 2p_5 |\psi|^6\right) dt. \tag{24}$$

While there are presumably additional conservation laws for this model, these preliminary ones are listed. These laws can be used to study soliton perturbation theory in presence of several perturbation terms to recover adiabatic dynamics of soliton parameters. These results will be eventually useful to address additional features such as collision-induced timing jitter, four-wave mixing and other such features.

6. Conclusions

In this work, we have proved that nonlinearly chirped solitary waves on a continuous-wave background can exist in a nonlinear metamaterial exhibiting higher-order effects such as pseudo-quintic nonlinearity and self-steepening effect. Using the traveling-wave method, we have presented two types of exact chirped gray solitary-wave solutions and the corresponding formation conditions in metamaterials. It is found that these chirped pulses possess a nontrivial phase structure which has two intensity dependent chirping terms, apart from the linear contribution. The stability of the solitary waves has been also demonstrated numerically with respect to finite perturbations of the additive white noise and perturbation of the amplitude. The results showed that the solutions we obtained are still stable under finite initial perturbations, such as amplitude and white noise. Due to their robust propagating nature, the newly found chirped gray solitary waves should be observed experimentally in metamaterials having a non-Kerr-type nonlinear response.

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