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THEORY OF NONLINEAR WAVES

Study of Optical Soliton Perturbation with Quadratic-Cubic Nonlinearity by Lie Symmetry and Group Invariance

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Abstract—Here, optical soliton perturbation with quadratic-cubic nonlinearity has been discussed by applying Lie symmetry and group invariants. The perturbation terms include third and fourth order dispersions in addition to self-steepening, intermodal dispersion and higher order dispersion effects. Using presented algorithms, Bright and dark soliton solutions are revealed.

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1. INTRODUCTION

In the field of mathematical physics, optical soliton perturbation has attracted researchers due to predominant applications to telecommunications industry. While several forms of non-Kerr laws of nonlinearity are visible and studied across the globe, this paper will study a particular form of such law. It is the quadratic-cubic (QC) nonlinearity that first appeared in the literature a few years ago. This form of optical fibers gained popularity ever since and several papers were published with such nonlinearity with the inclusion of Hamiltonian type perturbation terms [1–4].

Besides these popular integration schemes, there is a powerful analytical approach that is centuries old and is yet popular and still flourishing [5–8]. This is the Lie symmetry analysis. Several forms of nonlinear evolution equations have been fruitfully addressed using this method during the past [9–11]. This paper will address perturbed nonlinear Schrödinger's equation (NLSE) with QC nonlinearity using this technique. The perturbation terms include third order dispersion (3OD) and fourth order dispersion (4OD) as well as self-steepening effect along with intermodal dispersion and nonlinear dispersion. The nonlinear perturbative effects due to self-steepening and nonlinear dispersion appear with full nonlinearity in order to maintain the model on a generalized flavor. The detailed analysis for the extraction of bright and dark solitons are explored.

1.1. Governing Model

Governing model is perturbed NLSE with QC nonlinearity, in dimensionless form is given as [1–3]

$$iq_t + aq_{xx} + \Big(b_1|q| + b_2|q|^2\Big)q = i\Big[\alpha q_x - \gamma q_{xxx} - i\sigma q_{xxxx} + \lambda\big(|q|^{2m}q\big)_x + \theta\big(|q|^{2m}\big)_x q\Big], \tag{1}$$

where x and t are spatial and temporal variables; $q(x,t)$ the complex-valued wave profile; a, b_1, and b_2 represent the group velocity dispersion, quadratic and cubic nonlinearities, respectively; α the intermodal dispersion; γ and σ are 3OD and 4OD, respectively; θ and $m > 0$ denote the nonlinear dispersion and nonlinearity parameter, respectively. To avoid the formation of shock waves, the coefficient λ due to self-steepening is included. .

2. LIE GROUP ANALYSIS

The concept of continuous symmetry is formalized by Lie groups and, thus, become a part of the foundation of mathematics. The main motivation for investigating symmetries of differential equations is

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mapping of known solutions to other solutions, construction of invariant solutions, detection of linearizing transformation, etc. The key idea of Lie theory of symmetry analysis of differential equations relies on the invariance under a transformation of independent and dependent variables. In this section, we will perform Lie symmetry analysis [4–10] for Eq. (1). Consider

$$q(x,t) = u(x,t) + iv(x,t). \tag{2}$$

From (1) after using (2), the real and imaginary parts are

$$\begin{aligned} -v_t + au_{xx} + b_1 u\sqrt{u^2+v^2} + b_2 u(u^2+v^2) &= -\alpha v_x + \gamma v_{xxx} + \sigma u_{xxxx} \\ &- m(\lambda+\theta)(u^2+v^2)^{m-1}(2uvu_x + 2v^2 v_x) - \lambda(u^2+v^2)^m v_x, \end{aligned} \tag{3}$$

and

$$\begin{aligned} u_t + av_{xx} + b_1 v\sqrt{u^2+v^2} + b_2 v(u^2+v^2) &= \alpha u_x - \gamma u_{xxx} + \sigma v_{xxxx} \\ &+ m(\lambda+\theta)(u^2+v^2)^{m-1}(2u^2 u_x + 2uvv_x) + \lambda(u^2+v^2)^m u_x. \end{aligned} \tag{4}$$

One-parameter Lie group of transformations:

$$\begin{aligned} u^* &\Longrightarrow u + \epsilon\eta(x,t,u,v), \qquad v^* \Longrightarrow v + \epsilon\phi(x,t,u,v), \\ x^* &\Longrightarrow x + \epsilon\xi(x,t,u,v), \qquad t^* \Longrightarrow t + \epsilon\tau(x,t,u,v), \end{aligned} \tag{5}$$

where ϵ is an infinitesimal parameter. The associated vector field with the above group of transformations can be written as

$$V = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}. \tag{6}$$

Applying the second prolongation $pr^{(2)}V$ of V to Eqs. (3) and (4), we find that the coefficient functions ξ, τ, η, ϕ must satisfy the invariance condition

$$\begin{aligned} &-\phi^t + a\eta^{xx} + b_1\left[\eta\sqrt{u^2+v^2} + \frac{u(u\eta+v\phi)}{\sqrt{u^2+v^2}}\right] + b_2(3u^2\eta + 2uv\phi + \eta v^2) \\ &= -\lambda\Big[(u^2+v^2)^m\phi^x\Big] - \lambda\Big[2mv_x(u^2+v^2)^{m-1}(u\eta+v\phi)\Big] - m(\lambda+\theta) \\ &\times\Big[(u^2+v^2)^{m-1}(2uv\eta^x + 2uu_x\phi + 2u_x v\eta + 2v^2\phi^x + 4vv_x\phi)\Big] \\ &- 2m(m-1)(\lambda+\theta)(u^2+v^2)^{m-2}(u\eta+v\phi)(2uvu_x + 2v^2v_x) + \gamma\phi^{xxx} + \sigma\eta^{xxxx} - \alpha\phi^x, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\eta^t + a\phi^{xx} + b_1\left[(\phi\sqrt{u^2+v^2} + \frac{v(u\eta+v\phi)}{\sqrt{u^2+v^2}}\right] + b_2(3v^2\phi + 2uv\eta + \phi u^2) \\ &= \lambda\Big[(u^2+v^2)^m\eta^x\Big] + 2m\lambda u_x(u^2+v^2)^{m-1}(u\eta+v\phi) + m(\lambda+\theta)(u^2+v^2)^{m-1} \\ &\times(2u^2\eta^x + 4uu_x\eta + 2uv\phi^x + 2u\phi v_x + 2vv_x\eta) + 2m(m-1)(\lambda+\theta)(u^2+v^2)^{m-2} \\ &\times(u\eta+v\phi)(2u^2u_x + 2uvv_x) - \gamma\eta^{xxx} + \sigma\phi^{xxxx} + \alpha\eta^x, \end{aligned} \tag{8}$$

where η^t, ϕ^t, η^x, ϕ^x, η^{xx}, ϕ^{xx}, η^{xxx}, ϕ^{xxx}, η^{xxxx}, ϕ^{xxxx} are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables u_t, v_t, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}, u_{xxxx}, v_{xxxx}. From (7) and (8), by replacing differential coefficients to zero, the infinitesimals are determined. The following obtained differential equations in η, ϕ, ξ, and τ that need to be satisfied:

$$
\begin{aligned}
&\text{(i)}\ \ \xi_u = 0, \quad \xi_v = 0,\\
&\text{(ii)}\ \ \tau_x = 0, \quad \tau_u = 0, \quad \tau_v = 0,\\
&\text{(iii)}\ \ \phi_u = 0, \quad \phi_{vv} = 0,\\
&\text{(iv)}\ \ \eta_x = 0, \quad \eta_v = 0, \quad \eta_{uu} = 0,\\
&\text{(v)}\ \ -\eta_u + 4\xi_x = 0,\\
&\text{(vi)}\ \ -3\gamma\phi_{xv} + 3\gamma\xi_{xx} = 0,\\
&\text{(vii)}\ \ \tau_t - \phi_v = 0,\\
&\text{(viii)}\ \ -\phi_v + \xi_x = 0,\\
&\text{(ix)}\ \ -v^2\lambda\xi_x - v^2\xi_x\theta + v^2\phi_v\lambda + v^2\phi_v\theta + \lambda u\eta + 3\lambda v\phi + 2\theta v\phi, = 0,\\
&\text{(x)}\ \ 4m(m-1)(\lambda+\theta)v^2(u^2+v^2)^{m-2}(u\eta + v\phi) = 0,\\
&\text{(xi)}\ \ \alpha\phi_v + \gamma\xi_{xxx} - \alpha\xi_x + \xi_t = 0,\\
&\text{(xii)}\ \ 4m(m-1)(\lambda+\theta)uv(u^2+v^2)^{m-2}(u\eta + v\phi) = 0,\\
&\text{(xiii)}\ \ -2m(\lambda+\theta)(u^2+v^2)^{m-1}(uv\xi_x - uv\eta_u - u\phi - v\eta) = 0,\\
&\text{(xiv)}\ \ \sigma\xi_{xxxx} - a\xi_{xx} = 0,\\
&\text{(xv)}\ \ -\phi_v + 3\gamma\xi_x = 0,\\
&\text{(xvi)}\ \ \frac{b_1uv\phi}{\sqrt{u^2+v^2}} + \frac{b_1u^2\eta}{\sqrt{u^2+v^2}} + 2b_2uv\phi - \phi_t + b_1\eta\sqrt{u^2+v^2} + b_2\eta v^2 + 3b_2u^2\eta = 0.
\end{aligned}
\tag{9}
$$

The general solution of this large system provides following forms for the infinitesimal elements η, ϕ, ξ, and τ:

$$
\xi = C_1, \quad \tau = C_2, \quad \eta = 0, \quad \phi = 0, \tag{10}
$$

where C_1, C_2 are arbitrary constants. Obtained symmetries reduce by applying characteristic equations:

$$
\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta} = \frac{dv}{\phi}. \tag{11}
$$

For (1), solving (11), after applying symmetries, we get

$$
\zeta = x - wt, \quad q(x,t) = F(\zeta)\exp\big[iG(\zeta)\big]. \tag{12}
$$

3. REDUCED ODEs AND SOLITON SOLUTIONS TO NLSE

The suitable similarities obtained in (12) reduce (1) to the following ordinary differential equations:

$$
\begin{aligned}
&wF(\zeta)\,G'(\zeta) + aF''(\zeta) - aF(\zeta)\big[G'(\zeta)\big]^2 + b_1F(\zeta)^2 + b_2F(\zeta)^3 = -\lambda F(\zeta)^{2m+1}G'(\zeta) + 3\gamma F''(\zeta)\,G'(\zeta)\\
&\quad + 3\gamma F'(\zeta)\,G''(\zeta) + \gamma F(\zeta)\,G'''(\zeta) - \gamma F(\zeta)\big[G'(\zeta)\big]^3 - \alpha F(\zeta)\,G'(\zeta) + \sigma\Big\{F''''(\zeta) - 6F''(\zeta)\big[G'(\zeta)\big]^2\Big\}\\
&\quad + \sigma\Big\{-12F'(\zeta)\,G'(\zeta)\,G''(\zeta) - 3F(\zeta)\big[G''(\zeta)\big]^2 - 4F(\zeta)\,G'(\zeta)\,G'''(\zeta) + F(\zeta)\big[G'(\zeta)\big]^4\Big\}
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
&-wF'(\zeta) + 2aF'(\zeta)\,G'(\zeta) + aF(\zeta)\,G''(\zeta) = \lambda(2m+1)\,F(\zeta)^{2m}F'(\zeta) + 2m\theta F(\zeta)^{2m}F'(\zeta)\\
&\quad + \alpha F'(\zeta) - \gamma F'''(\zeta) + 3\gamma F'(\zeta)\big[G'(\zeta)\big]^2 + 3\gamma F(\zeta)\,G'(\zeta)\,G''(\zeta) + \sigma\big[4F'''(\zeta)\,G'(\zeta) + 6F''(\zeta)\,G''(\zeta)\big]\\
&\quad + \sigma\Big\{4F'(\zeta)\,G'''(\zeta) + F(\zeta)G''''(\zeta) - 4F'(\zeta)\big[G'(\zeta)\big]^3 - 6F(\zeta)\big[G'(\zeta)\big]^2G''(\zeta)\Big\}.
\end{aligned}
\tag{14}
$$

Now, Eqs. (13) and (14) will be used to retrieve soliton-like solutions of NLSE in the following subsections.

3.1. Bright Soliton Solution to NLSE

By taking the following forms of $F(\zeta)$ and $G(\zeta)$, we get bright soliton solutions to NLSE (1):

$$F(\zeta) = A\,\mathrm{sech}(\zeta)^p, \quad G(\zeta) = B\zeta, \tag{15}$$

where A, B are the amplitude and width of the soliton. Balancing principle is applied to find p. By equating the exponents $(2m+1)p$ and $p+2$ and using (15) in (13), (14), we get $p=1/m$. Also the solution of (1) becomes

$$q(x,t) = A\,\mathrm{sech}\big[t(B^2\gamma-\alpha+\gamma)-x\big]\exp\Big\{iB\big[-t(B^2\gamma-\alpha+\gamma)+x\big]\Big\}, \tag{16}$$

where $a=2B\gamma$, $b_1=0$, $b_2=-\dfrac{B(A^2\lambda+2\gamma)}{A^2}$, $\sigma=0$, and $\theta=-\dfrac{3(A^2\lambda+2\gamma)}{2A^2}$.

3.2. Dark Solitons and Other Solutions

On substituting $G(\zeta)=B\zeta$ in (13), (14) and equating the coefficients equal to zero of linearly independent functions, we have

$$\sigma=\frac{\gamma}{4B}, \quad \lambda=\frac{-2\theta}{3}, \quad \alpha=-2B^2\gamma+2aB-w, \tag{17}$$

Then reduced ODEs (13) and (14) provides the following solutions to NLSE for $m=1$:

$$q(x,t)=\left[\left\{\tanh\left[-\frac{1}{2}B(-wt+x)+C_1\right]\right\}^2 C_4-C_4\right]\exp\big[iB(-wt+x)\big], \tag{18}$$

with $a=\dfrac{3\gamma B}{2}$, $b_1=\dfrac{15B^3\gamma}{8C_4}$, and $b_2=\dfrac{B(45B^2\gamma+16C_4^2\theta)}{24C_4^2}$;

$$q(x,t)=\frac{1}{4}\frac{B^2(5\gamma B-4a)\exp\big[i(-wt+x)B\big]}{b_1}, \tag{19}$$

with $b_2=\dfrac{2\theta B}{3}$;

$$q(x,t)=\left[\frac{35\big\{\tanh\big[-(i/6)(-wt+x)B+C_1\big]\big\}^4\gamma B^3}{216b_1}\right.$$
$$\left.-\frac{35\big\{\tanh\big[-(i/6)(-wt+x)B+C_1\big]\big\}^2\gamma B^3}{108b_1}+\frac{35\gamma B^3}{216b_1}\right]\exp\big[i(-wt+x)B\big], \tag{20}$$

with $b_2=\dfrac{2\theta B}{3}$ and $a=\dfrac{41\gamma B}{36}$;

$$q(x,t)=\left[\left\{\tanh\left[-\frac{1}{20}\sqrt{10}B(-tw+x)+C_1\right]\right\}^2 C_4+C_3\right]\exp\big[iB(-tw+x)\big], \tag{21}$$

with $a=\dfrac{5\gamma B}{4}$, $b_1=-\dfrac{23B^3\gamma}{800C_3}$, and $b_2=\dfrac{B(23B^2\gamma+1600C_3^2\theta)}{2400C_3^2}$;

$$q(x,t)=C_4\tanh\left[-\frac{1}{2}\sqrt{2}B(-wt+x)+C_1\right]\exp\big[iB(-wt+x)\big], \tag{22}$$

with $a=\dfrac{C_4^2(2\theta B-3b_2)}{3B^2}$, $b_1=0$, and $\gamma=0$.

4. CONCLUSIONS

The perturbed NLSE with QC nonlinearity was studied in this paper with Lie symmetry and group invariants. Bright and dark soliton solutions are retrieved. The perturbation terms included 3OD and 4OD. It must be noted that traveling wave hypothesis and the method of undetermined coefficients fail to retrieve soliton solutions to the perturbed NLSE with these two perturbation terms included. Thus, this integration scheme has a true edge over the other two. Although semi-inverse variational principle retrieved bright soliton solution to the model, it must be noted that this method, although analytical, does not yield an exact soliton solution [1]. Thus Lie symmetry analysis still stands strong even today. The results of this paper pave way to further research activities in future.

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